Harmonic Analysis Class Notes

Contents

0.1. Notation

- $C(X,Y)$: The set of continuous functions from X to Y
- $C^k(X,Y)$: The set of functions from X to Y that are continuous ($k=0$), continuously k-times differentiable ($k\in\mathbb{N}$), smooth ($k=\infty$), or analytic ($k=\omega$), if well-defined between spaces X and Y.
- $C_c^k(X,Y)$: The set of compactly supported functions contained in $C^k(X,Y)$.
- $C_0^k(X,Y)$: If Y is some norm space, the set of functions $f\in C^k(X,Y)$ that vanish at infinity, i.e. for all $\varepsilon>0$, there exists some compact $K\subseteq X$ such that for $x\notin X$, $\|f\|_Y<\varepsilon$
- $C(X)=C(X,\mathbb{C})$
- $C^k(X)=C^k(X,\mathbb{C})$
- $C_c(X)=C_c^k(X,\mathbb{C})$
- $C_0(X)=C_0^k(X,\mathbb{C})$
- $L^p(X,\mu)$, $L^p(X)$: For $0<p<\infty$, the space of measurable functions $f:X\to\mathbb{C}$ such that

$$\|f\|_{L^p(X,\mu)}:=\Big(\int_X|f(x)|^p d\mu(x)\Big)^{1/p}<\infty.$$

  If $p=\infty$, this is the space of measurable functions $f:X\to\mathbb{C}$ such that

$$\|f\|_{L^\infty(X,\mu)}:=\inf\{C>0:|f(x)|<C\ \mu-a.e.\}$$

  If μ is understood, we will simply write $L^p(X)$ for $L^p(X,\mu)$.
- $\mathcal{F}(f)=\hat{f}$: The Fourier transform of a function f.
- $\mathcal{F}(\mu)=\hat{\mu}$: The Fourier-Steiljes Transform of a Radon measure of bounded total variation, μ.
- $A+B$, $A-B$: For any group $(G,+)$ with $-x$ denoting the inverse of $x\in G$, for $A,B\subseteq G$, $A+B=\{a+b:a\in A,b\in B\}$ is called the **sumset** of A and B, $-B=\{-b:b\in B\}$ and $A-B=A+(-B)$ is the **difference set** of A and B.
- $A+x$: For any group $(G,+)$, $x\in G$, and $A\subseteq G$, $A+x=\{a+x:a\in A\}$.

0.2. Introduction

Harmonic Analysis is a very broad subject with several different facets such as

- The study of certain qualitative or quantitative properties of classes of functions and how the properties change when one applies various operators (Fourier Transform, Hilbert Transform, Convolution, etc). For instance if $f,g\in L^2(\mathbb{R}^n)$ it can be shown that $\|f*g\|_{L^\infty(\mathbb{R}^n)}\le\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$.
- The study of the eigenvalues and eigenfunctions of the Laplacian on manifolds (or other domains), usually to use properties of these to study more complicated functions (Spherical harmonics, trigonometric polynomials).
- Representation of functions as the superposition of "basic waves" (Spherical harmonics, Fourier Series, Wavelets).
- The study of functions using group actions/ symmetries such at translations or rotations (Harmonic Analysis on Homogeneous spaces, Fourier Analysis on locally compact groups)

Another way to describe Harmonic Analysis is as a generalization of Fourier Analysis, which could be describe as the study of representing or approximating functions by trigonometric functions. The key ingredient for this is known as the Fourier transform. This can be defined on compact topological groups and locally compact Abelian groups (which will be defined later), but the more well known domains are finite Abelian groups (in which the Fourier transform is known as the Discrete Fourier Transform), the circle $\mathbb{T}=[0,1)$ (where one has Fourier Series), and Euclidean space $\mathbb{R}$ (where the Fourier transform is called the Fourier transform, oddly enough).

For a function $f \in L^1(\mathbb{T})$, the Fourier series of f is given by

$$f(t) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi int}, \tag{0.2.1}$$

where

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx. \tag{0.2.2}$$

Alternatively, this function can be expressed as

$$f(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nt) + b_n \sin(2\pi nt), \tag{0.2.3}$$

with appropriate coefficients a_n and b_n. This is the decomposition of a "signal" f into wave functions of frequencies n. Such decompositions have a wide range of applications, with some of the earliest being to find solutions to differential equations.

- Wave Equation: Let $u(x,t)$ on $\mathbb{T} \times [0,\infty)$ satisfy $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and $u(0,t) = u(1,t) = 0$. We wish to determine what u could be, other than the trivial solution $u = 0$. Assuming that $u(x,t) = g(x)h(t)$, we have

$$\frac{g''(x)}{g(x)} = \frac{h''(t)}{h(t)} = \lambda \tag{0.2.4}$$

  for some λ. Thus $g''(x) - \lambda g(x) = 0$ and $g(0) = g(1) = 0$. The solutions to this would be of the form $g(x) = b_n \sin(2\pi nx)$ if $\lambda = -4\pi^2 n^2$. Plugging in for λ, we have the solutions for $h''(t) + 4\pi^2 n^2 h(t) = 0$ are of the form $a_n \sin(2\pi nt) + c_n \cos(2\pi nt)$, so our general solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \sin(2\pi nt) + c_n \cos(2\pi nt)\Big) b_n \sin(2\pi nx). \tag{0.2.5}$$

  This wave equation describes the motion of a string that is held in place at both ends, with $u(x,t)$ being the height of the string at x and time t.
- Heat equation: Here, $u(x,t)$ on $\mathbb{T} \times [0,\infty)$ satisfies $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$ for some function $f \in L^1(\mathbb{T})$, and $u(0,t) = u(1,t)$.

  In this case, one gets as a general solution

$$u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{-4\pi^2 n^2 t}e^{2\pi inx}. \tag{0.2.6}$$

  This heat equation describes the way the distribution along a ring with, with $u(x,t)$ being a measurement of heat at x and time t.

While such expressions are useful, this leads to several natural questions. When do these series converge, and in what sense? Are there conditions on a function f we can place so that $\hat{f}(n) \to 0$ as $|n| \to \infty$? What else can we say about the Fourier coefficients of a function f based on properties of f?

The Fourier transform of a function f on $\mathbb{R}$ is given by

$$\hat{f}(s) = \int_{\mathbb{R}} f(t)e^{-2\pi ist}dt. \tag{0.2.7}$$

The Fourier transform in this sense takes a function of time and sends it to a function of frequency. As we will show, various operations (differentiation, convolution, etc) of functions in one domain have a corresponding operation in the other. Generally speaking, the operation in one of the domains is easier than the corresponding operation in the other, allowing one to simplify the study of more complicated functions.

CHAPTER 1

Fourier Analysis on Locally Compact Abelian Groups

We can generalize the theory used for classical spaces, connecting them and shedding more light as to what is happening. The work here is mostly based off of the [**Rud90, Fol15**]. We recommend [**Rud90**] for more exposition with relatively few requirements, and [**Fol15**] for building this theory with Representation Theory. In both cases, Gelfand Theory and Banach Algebras play a role. This all makes for more general theory than we need for this course, and so our methods may differ.

1.1. Locally Compact Abelian Groups

A **topological group** G is a group G with a topology such that the group operation $* : G\times G \to G$ and inverse operation $\cdot^{-1} : G \to G$ are continuous. We call G **locally compact** if the topology is **Hausdorff** (i.e. for every distinct $x, y \in G$, there exist disjoint open sets $A, B \subset G$ such that $x \in A$ and $y \in B$), and **locally compact** (for every $x \in G$, there exist a **compact neighborhood** B of x, i.e. an open set A and a compact set B such that $x \in A \subseteq B$).

While the theory of Fourier Analysis can also be built for compact groups, such as $\mathrm{SO}(d)$, this requires representation theory, and will not be covered in the scope of this course. Instead, we shall focus on locally compact Abelian groups.

Example 1.1.1. *Here are some examples of locally compact Abelian groups*

1. *The real numbers, $\mathbb{R}$, under addition and Euclidean topology.*
2. *The integers, $\mathbb{Z}$, under addition and the discrete topology*
3. *The torus, $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z} \simeq [0, 1)$ (with 0 identified with 1), under addition $(\mod 1)$ and with the quotient topology.*
4. *The cyclic group, $\mathbb{Z}_n$, under addition $(\mod n)$ and with the discrete topology.*
5. *The set of positive real numbers, $\mathbb{R}_+$, under multiplication and Euclidean topology.*
6. *The rational numbers $\mathbb{Q}$ under addition and the discrete topology.*
7. *The field of p-adic numbers, $\mathbb{Q}_p$, under addition and with the usual p-adic topology.*
8. *Any Abelian group under the discrete topology.*
9. *The Adele ring.*
10. *If G is locally compact Abelian group and H is a closed subgroup, both H and G/H.*
11. *Finite direct sums of locally compact Abelian groups with the product topology.*
12. *The set of all sequences $\{a_n\}_{n=1}^{\infty}$, with $a_n \in \mathbb{Z}_2$, with addition performed coordinatewise, i.e. $\{a_n\}_{n=1}^{\infty} + \{b_n\}_{n=1}^{\infty} = \{a_n + b_n (\mod 2)\}_{n=1}^{\infty}$, and with topology making the map $\{a_n\}_{n=1}^{\infty} \mapsto 2\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ a homeomorphism from the group to the classical Cantor set.*

For the rest of this section, we will assume that G is a locally compact Abelian group, with binary operation $+$, and $-x$ as the inverse of $x \in G$.

1.2. Haar Measure

Definition 1.2.1. *A **Radon measure** μ on a locally compact Hausdorff space G is a Borel measure which is*

(1) ***locally finite***: *for all $x \in G$, there is some neighborhood U of x such that $\mu(U) < \infty$.*
(2) ***outer regular*** *on Borel sets: if $E \subseteq G$ is Borel, then*

$$\sigma(E) = \inf\{\sigma(U) : E \subseteq U \subseteq G,\ U \text{ open}\}.$$

(3) ***inner regular*** *on open sets: if $U \subseteq G$ is open, then*

$$\sigma(U) = \sup\{\sigma(K) : K \subseteq U,\ K \text{ compact}\}.$$

If there exist Radon measures $\mu_1, \mu_2, \mu_3, \mu_4$ such that $\operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2) = \operatorname{supp}(\mu_3) \cap \operatorname{supp}(\mu_4) = \emptyset$ *and*

$$\mu := \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

is well defined on Borel sets of G, we call μ a ***complex Radon measure****.*

Definition 1.2.2. *A* ***Haar measure*** *σ on a locally compact Abelian group G is a nonzero Radon measure that is translation invariant, i.e.*

$$\sigma(U) = \sigma(U + x) \quad \text{for all measureable } U \subseteq G \text{ and } x \in G. \tag{1.2.1}$$

Theorem 1.2.3 (Haar's Theorem). *Let G be a locally compact Abelian group. Then there exists a Haar measure σ on G. This measure is unique up to a constant, i.e. if σ and ν are both Haar measures on G, then there is some $c \in (0, \infty)$ such that $c\nu = \sigma$.*

A general discussion of Haar measures can be found in [**Rud90**][Chp 1.1] or [**Fol15**][Chp 2.2], with the latter containing a proof of Theorem 1.2.3 (Haar's Theorem). Left and right Haar measures can be defined for general locally compact groups, and there is a more general version of Haar's Theorem that gives existence and uniqueness of each of these measures. In general, left and right Haar measures are not the same. However, under certain conditions, such as G being Abelian or compact, one finds a Haar measure that is both left and right translation-invariant. In this case, the Haar measure is also invariant under inversion, which is a very useful property for getting Fourier analytic results. Generally, one takes a specific Haar measure, usually with $\sigma(G) = 1$ if G is compact, and refers to it as *the* Haar measure. We will collect a few properties of the Haar measure:

Proposition 1.2.4. *Let G be a locally compact Abelian group, and σ its Haar measure. Then*
(1) *If $E \subseteq G$ is a Borel set and $x \in G$, $\sigma(E) = \sigma(-E) = \sigma(x + E) = \sigma(E + x)$.*
(2) *If $K \subseteq G$ is compact, then $\sigma(K)$ is finite (follows from local finiteness and countable additivity of Radon measures).*
(3) *If $U \subseteq G$ is non-empty, then $0 < \sigma(U)$ (follows from translation-invariance and inner regularity of Radon measures).*

The Haar measure of any nonempty open set in G is always positive (or else the translation invariance would mean $\sigma(G) = 0$).

Example 1.2.2. *Here are some examples of Haar measures for certain groups*
(1) $\mathbb{R}$*: Lebesgue measure*
(2) $\mathbb{T}$*: Normalized Lebesgue measure on* $[0, 1)$
(3) *Any discrete group: The counting measure (possibly normalized if compact)*
(4) $\mathbb{R}_+$*:* $\frac{dx}{|x|}$
(5) *Direct sums of Abelian groups: product measure of corresponding Haar measures.*

Unless otherwise noted, for a locally compact Abelian group G, we will denote the Haar measure as σ and $L^p(G)$ as the L^p space on G corresponding to σ.

We also note that for general locally compact Abelian groups G, we do not assume that G is σ-compact and so the Haar measure may not be σ-finite. This lack of σ-finiteness occurs, for instance, whenever G is uncountable and has the discrete topology. This could mean there are cases where we would be prevented from using certain results, such as Fubini's Theorem. However, in the cases where we use it in this chapter (involving function in L^p spaces for $1 \leq p < \infty$) this ends up not being a problem, as shown by the discussion in Chapter 2.3 in [**Fol15**]. In particular, we will be implicitly using Proposition 2.0.1

1.2.1. Properties of L^p Spaces and Convolution.

PROPOSITION 1.2.5. *For all $p \in [1, \infty)$, the set of compactly supported, complex-valued, continuous functions on G, $C_c(G)$, is dense in $L^p(G)$.*

This can be found in [**Rud87**][Thm 3.14].

LEMMA 1.2.6. *Let $f \in L^p(G)$ and for $y \in G$, define $L_y f(x) = f(y + x)$. The map $y \mapsto L_y f$ is continuous.*

PROOF. Assume that $f \in C_c(G)$, and let $A = \operatorname{supp}(f)$ and B be a compact neighborhood of 0. Compact support means that f is uniformly continuous, so there is some neighborhood V_ε of 0 such that if $x - z \in V_\varepsilon$, then $|f(x) - f(z)| < \varepsilon$, and let $C = B \cap V_\varepsilon$. For $y \in C$, if $f(y + x) - f(x) \neq 0$, then $y + x \in A$ or $x \in A$, so since $0 \in C$, $x \in A - C = \{a - b : a \in A, b \in C\}$, which has compact closure. We see that $f(y + x) - f(x)$ vanishes outside of $A - C$, and $y + x - x = y \in V_\varepsilon$ so

$$\Big(\int_G |f(y + x) - f(x)|^p d\sigma(x) \Big)^{\frac{1}{p}} = \Big(\int_{A-C} |f(y + x) - f(x)| d\sigma \Big)^{\frac{1}{p}} \leq \varepsilon \sigma(A - C)^{\frac{1}{p}}.$$

Due to the density of $C_c(G)$ in $L^p(G)$, we have that for any $\varepsilon > 0$ and $f \in L^p(G)$, there exists some $g \in C_c(G)$ such that $\|f - g\|_{L^p(G)} < \varepsilon$, so for any $y \in G$,

$$\|L_y f - f\|_{L^p(G)} \leq \|L_y(f - g)\|_{L^p(G)} + \|L_y g - g\|_{L^p(G)} + \|f - g\|_{L^p(G)} < 2\varepsilon + \|L_y g - g\|_{L^p(G)}.$$

$\square$

We define convolution on G

$$(f * g)(y) = \int_G f(y - x) g(x) d\sigma(x). \tag{1.2.3}$$

Loosely speaking, convolutions tend to preserve many "nice" properties of the functions involved, or even improve upon them.

EXAMPLE 1.2.4. *Let $f(x) = \mathbb{1}_{[-1,1]}(x)$ on $\mathbb{R}$. Then we have*

$$\begin{aligned} (f * f)(x) &= \int_{\mathbb{R}} f(x - y) f(y) dy \\ &= \int_{[x-1,x+1]\cap[-1,1]} dy \\ &= \begin{cases} 0 & |x| > 2 \\ 2 - |x| & |x| \leq 2 \end{cases}. \end{aligned}$$

So we have gained continuity.

Conceptually, by convolving f with "good" functions g, we can replace $f(x)$ with a function that takes an "average" around x, particularly in the case that g is nonnegative and $\int g = 1$. If g is is smooth, this is known as mollifying.

Convolution leads to continuity and boundedness in a much more general situation.

Lemma 1.2.7. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f * g\|_{L^\infty(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^q(G)} \tag{1.2.5}$$

and $f * g \in C_0(G)$.

Proof. Let $x \in G$. Then we have, by Hölder's inequality

$$\begin{aligned} |(f * g)(x)| &= \Big| \int_G f(x-y) g(y) d\sigma(y) \Big| \\ &\leq \int_G |f(x-y)| \, |g(y)| d\sigma(y) \\ &\leq \Big(\int_G |f(x-y)|^p d\sigma(y) \Big)^{1/p} \Big(\int_G |g(y)|^q d\sigma(y) \Big)^{1/q} \end{aligned}$$

and the invariance of the Haar measure under translation and inversion give us (1.2.5).

Now, suppose that $x, a \in G$. Then by the invariance of the Haar measure under translation and inversion and Hölder's inequality

$$\begin{aligned} |(f * g)(x-a) - (f * g)(x)| &= \Big| \int_G f(x-a-y) g(y) d\sigma(y) - \int_G f(x-y) g(y) d\sigma(y) \Big| \\ &= \Big| \int_G f(x-y) g(y-a) - f(x-y) g(y) d\sigma(y) \Big| \\ &\leq \int_G |f(x-y)| |g(y-a) - g(y)| d\sigma(y) \\ &\leq \|f\|_{L^p(G)} \|L_{-a} g - g\|_{L^q(G)}. \end{aligned}$$

The continuity Lemma 1.2.6, then tells us that $f * g$ is continuous. Thus, if $f, g \in C_c(G)$, $f * g \in C_c(G)$.

From 1.2.5, we may choose sequences of functions $f_n, g_n \in C_c(G)$ such that $f_n \to f$ in $L^p(G)$ and $g_n \to g$ in L^q. Using the triangle inequality and Hölder's inequality, we find

$$\begin{aligned} \|f * g - f_n * g_n\|_{L^\infty(G)} &\leq \|f * g - f * g_n\|_{L^\infty(G)} + \|f * g_n - f_n * g_n\|_{L^\infty(G)} \\ &\leq \|f\|_{L^p(G)} \|g - g_n\|_{L^q(G)} + \|f - f_n\|_{L^p(G)} \|g\|_{L^q(G)}, \end{aligned}$$

so $f_n * g_n$ converges uniformly to $f * g$. Since each $f_n * g_n$ as compact support, and $f * g \in C(G)$, $f * g$ must vanish at infinity. □

One can quickly check that convolution preserves functions being compactly supported. If $\operatorname{supp}(f) = K \subseteq G$ and $\operatorname{supp}(g) = M \subseteq G$ then $\operatorname{supp}(f * g) \subseteq K + M$, so if K and M are compact, so is $\operatorname{supp}(f * g)$.

Corollary 1.2.8. *If $f, g \in C_c(G)$, $f * g \in C_c(G)$ and $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.*

Proof. Since $f, g \in L^2(G)$, $f * g \in C_0(G)$ by Lemma 1.2.7. Now, if $A = \operatorname{supp}(f)$ and $B = \operatorname{supp}(g)$, then

$$(f * g)(x) = \int_G f(x-y) g(y) d\sigma(x) = \int_B f(x-y) g(y) d\sigma(x).$$

Thus, $(f * g)(x) \neq 0$ only if $x - y \in A$ for some $y \in B$, which means $x \in A + y \subset A + B$. □

Finally, convolution preserves the integrability of functions.

PROPOSITION 1.2.9. *If $f, g \in L^1(G)$, then*

$$\|f * g\|_{L^1(G)} \leq \|f\|_{L^1(G)}\|f\|_{L^1(G)}$$

PROOF. Using translation invariance of the Haar measure and Fubini's Theorem, we have

$$\begin{aligned}\int_G |(f*g)(y)|d\sigma(y) &\leq \int_G \int_G |f(y-x)| \cdot |g(x)|d\sigma(x)d\sigma(y)\\ &= \int_G |g(x)| \int_G |f(y-x)|d\sigma(y)d\sigma(x)\\ &= \int_G |g(x)|d\sigma(x) \int_G |f(y)|d\sigma(y).\end{aligned}$$

□

This means $L^1(G)$ is a Banach algebra with addition and convolution (acting as the multiplication operation). We note that Proposition 1.2.9 can generalized to show that the convolution of an L^1 function and L^p function, for $p \in [1, \infty]$, is itself an L^p function (See [**Fol15**][Proposition 2.40]).

1.2.2. Approximate Identities. When G is discrete, the function δ_0, defined by $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$, is the identity element of the algebra $L^1(G)$: $(f * \delta_0)(x) = (\delta_0 * f)(x) = f(x)$. Note that $\int_G f(x)\delta_0(x)d\sigma(x) = (f * \delta_0)(0) = f(0)$. When G is not discrete, there is a measure that acts this way, but no such function. It is helpful to remain in the realm of functions, but to also have something to substitute for an identity element, so we instead use an *approximate identity.*

DEFINITION 1.2.10. *An **approximate identity** $\{\psi_U\}_{U\in\mathcal{U}}$, for some local basis $\mathcal{U}$ for 0 (see Definition 1.3.2), is a family of functions $\psi_U \in L^1(G)$, such that*

(1) $\operatorname{supp}(\psi_U) \subseteq U$ *is compact,*
(2) $\psi_U(x) \geq 0$ *for all $x \in G$,*
(3) $\int_G \psi_U(x)d\sigma(x) = 1$.

We will often abuse notation, and call ψ_U an approximate identity for convenience.

We can place a partial ordering on $\mathcal{U}$ by saying $U_1 \leq U_2$ if $U_1 \subseteq U_2$. We note that $0 \in U_1 \cap U_2$, so there is always some $U_3 \in \mathcal{U}$ such that $U_3 \subseteq U_1 \cap U_2$. Thus, for any $U_1, U_2 \in \mathcal{U}$, there is some $U_3 \in \mathcal{U}$ such that $U_3 \leq U_1, U_2$. If A is some normed vector space, $h \in A$, and $P : \{\psi_U\}_{U\in\mathcal{U}} \to A$, we say that

$$\lim_{U\to\{0\}} P(\psi_U) = h$$

if, for all $\varepsilon > 0$, there is some $U \in \mathcal{U}$ such that for all $V \leq U$, $\|P(\psi_V) - h\|_A < \varepsilon$.

In other words, an approximate identity is a *net.* However, if G is first countable, which is the case for most spaces of interest, such as $\mathbb{R}^d$, $\mathbb{T}^d$, and discrete spaces, we may instead take a sequence of neighborhoods $\{U_j\}_{j\in\mathbb{N}}$, with $0 \in U_{j+1} \subset U_j$.

We also note that the definition of an approximate identity means that for any neighborhood V of 0,

$$\lim_{U\to\{0\}} \int_{G\setminus V} \psi_U(x)d\sigma(x) = 0,$$

so the total mass is eventually concentrated near/at 0.

EXAMPLE 1.2.6. *Here are some examples of approximate identities:*

(1) *Let $B(0,r)$ be the balls of radius r and centered at the origin in $\mathbb{R}^n$. These form a local basis of 0. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative integrable function such that $\operatorname{supp}(\phi)$ is compact and contained in $B(0,1)$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Define $\phi_r(x) = r^{-n}\phi(\frac{x}{r})$ for $r > 0$. We quickly see that all ϕ_r are nonnegative and compactly supported on $r\operatorname{supp}(\phi) \subset B(0,r)$. From a change of variables $y = \frac{x}{r}$, we see as well that*

$$\int_{\mathbb{R}^n} \phi_r(x)dx = \int_{\mathbb{R}^n} r^{-n}\phi(\frac{x}{r})dx = \int_{\mathbb{R}^n} \phi(y)dy = 1. \tag{1.2.7}$$

(2) *There exists some compact neighborhood K of 0. For each U in the local basis, let $V_U = K \cap U$, which is a neighborhood of 0 with compact closure. Define $\psi_U(x) = \frac{1}{\sigma(V_U)}\mathbb{1}_{V_U}(x)$.*

(3) *By Urysohn's Lemma (see* [**Rud87**]*[Lemma 2.12] for the relevant version), for each U in the local basis, there exists a $\psi_U \in C_c(G)$ such that $\psi_U(x) = 1$ on $\overline{V_U}$ (from the example above), $\operatorname{supp}(\psi_U) \subseteq U$ and $\psi_U(G) = [0,1]$.*

THEOREM 1.2.11 (Minkowski's Inequality for Integrals). *Let (X, M, μ) and (Y, N, ν) be σ-finite measure spaces, and f be an $(M \otimes N)$-measureable function on $X \times Y$. If $1 \le p \le \infty$, $f(\cdot, y) \in L^p(G, \mu)$ for a.e. y, and the function defined by $y \mapsto \|f(\cdot, y)\|_{L^p(X,\mu)}$ is in $L^1(Y, \nu)$, then*

$$\Big\| \int_Y f(\cdot, y)d\nu(y)\Big\|_{L^p(X,\mu)} \le \int_Y \|f(\cdot, y)\|_{L^p(X,\mu)}d\nu(y).$$

This can be found in [**Fol99**][Thm 6.19].

THEOREM 1.2.12. *Let ψ_U be an approximate identity. Then*

$$\lim_{U\to\{0\}} \|f * \psi_U - f\|_{L^p(G)} = 0$$

for $1 \le p < \infty$ and $f \in L^p(G)$, or $p = \infty$ and f uniformly continuous.

PROOF. Since $\int_G \psi_U(x)d\sigma(x) = 1$, we have

$$f * \psi_U - f = \int_G \psi_U(y)\big(L_{-y}f(x) - f(x)\big)d\sigma(y).$$

If $1 \le p < \infty$ and $g \in L^p(G)$, then σ is σ-finite on $\operatorname{supp}(g)$, by Proposition 2.0.1, and since $\operatorname{supp}(\psi_U)$ is compact, the measures defined by $\mathbb{1}_{\operatorname{supp}(g)}(z)d\sigma(z)$ and $\psi_U(y)d\sigma(y)$ are σ-finite on G. By Minkowski's inequality (not the one for integrals), we know that $L_{-y}f - f \in L^p(G)$, and the continuity of translation (from Lemma 1.2.6) and inversion on G tells us that for we have that for U a sufficiently small neighborhood of 0

$$\int_G \Big|\|L_{-y}f - f\|_{L^p(G)}\psi_U(y)\Big|d\sigma(y) < \infty,$$

so we may use Theorem 1.2.11 (Minkowski's inequality for Integrals), giving us

$$\begin{aligned}\|f * \psi_U - f\|_{L^p(G)} &= \Big\|\int_G \psi_U(y)\big(L_{-y}f - f\big)d\sigma(y)\Big\|_{L^p(G)} \\ &\le \int_G \|L_{-y}f - f\|_{L^p(G)}\psi_U(y)d\sigma(y) \\ &\le \sup_{y\in U} \|L_{-y}f - f\|_{L^p(G)}.\end{aligned}$$

Continuity then tells us this goes to zero as $U \to \{0\}$.

If $p = \infty$ and f is then

$$\sup_{x\in G}\Big|\int_G \psi_U(y)\big(L_{-y}f(x)-f(x)\big)d\sigma(y)\Big| \leq \sup_{x\in G}\int_G |\psi_U(y)|\cdot|L_{-y}f(x)-f(x)|d\sigma(y)$$
$$\leq \int_G |\psi_U(y)|\sup_{x\in G}|L_{-y}f(x)-f(x)|d\sigma(y),$$

and the uniform continuity of f finishes the proof. □

Between this result and our discussion on convolution, we can now construct an explicit family of functions which allow us to approximate a function f, while gaining other useful properties, such as continuity.

We required ψ_U to be compactly supported in order guarantee that we met the σ-finiteness conditions of Theorem 1.2.11 (Minkowski's Inequality for Integrals). However, if σ is already σ-finite, we can actually loosen our restrictions.

Definition 1.2.13. *If σ is σ-finite, then we define a family of $L^1(G)$ functions ψ_ε to be an **approximate identity**, as $\varepsilon \to 0$, if*

(1) *There exists some $c > 0$ such that $\|\psi_\varepsilon\|_{L^1(G)} \leq c$ for all $\varepsilon > 0$,*
(2) *$\int_G \psi_\varepsilon(x)d\sigma(x) = 1$ for all $\varepsilon > 0$,*
(3) *for any neighborhood U of 0, we have $\int_{G\setminus U}|\psi_\varepsilon(x)|d\sigma(x) \to 0$ as $\varepsilon \to 0$.*

Example 1.2.8. *With this broader definition, we have that the following examples of approximate identities (see* [**Gra10**]*[Examples 1.2.16 and 1.2.18]):*

(1) *On $\mathbb{R}$, let $P(x) = \frac{1}{\pi(x^2+1)}$ and $P_\varepsilon = \varepsilon^{-1}P(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. The function P_ε is called a **Poisson kernel** and these form an approximate identity.*
(2) *On $\mathbb{T}$, for $N \in \mathbb{N}$, let*

$$F_N(t) = \sum_{j=-N}^{N}\Big(1-\frac{|j|}{N+1}\Big)e^{2\pi ijt} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2. \tag{1.2.9}$$

*The function F_N is called a **Fejér kernel**, and these form an approximate identity.*

1.3. Characters and Dual Group

Definition 1.3.1. *A **character** on a locally compact Abelian group G is a continuous homomorphism $\xi : G \to S^1 := \{e^{2\pi ix} : x \in [0,1)\}$. We call the set of all characters, $\hat{G}$, the **dual group** of G.*

Example 1.3.1. *Here we provide some examples of the duals of locally compact Abelian groups.*

(1) *$\hat{\mathbb{R}} \simeq \mathbb{R}$, with $\xi(x) = e^{2\pi i\xi x}$*

If $\xi \in \hat{\mathbb{R}}$, we have $\xi(0) = 1$, so there exists $a > 0$ such that $\int_0^a \xi(t)dt \neq 0$. Then

$$\xi(x)\int_0^a \xi(t)dt = \int_0^a \xi(t+x)dt = \int_x^{a+x}\xi(t)dt,$$

so (by the fundamental theorem of calculus), ξ is differentiable, with

$$\xi'(x) = \frac{\xi(a+x)-\xi(x)}{\int_0^a \xi(t)dt} = c\xi(x), \quad c : \frac{\xi(a)-1}{\int_0^a \xi(t)dt}.$$

Therefore, $\xi(x) = e^{cx}$, and since $|\xi(x)| = 1$, we must have $c = 2\pi iy$ for some $y \in \mathbb{R}$.

(2) $\hat{\mathbb{T}} \simeq \mathbb{Z}$, *with* $\xi(x) = (e^{2\pi i x})^{\xi}$.
Since $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$, *the characters of* $\mathbb{T}$ *are the characters of* $\mathbb{R}$ *which are* 1 *on* $\mathbb{Z}$.
(3) $\hat{\mathbb{Z}} \simeq \mathbb{T}$, *with* $\xi(x) = (e^{2\pi i \xi})^{x}$.
If $\xi \in \hat{Z}$, *then with* $\alpha = \xi(1)$, $\xi(n) = \xi(1)^n = \alpha^n$.
(4) $\hat{\mathbb{Z}}_n \simeq \mathbb{Z}_n$, *with* $\xi(x) = e^{2\pi i \xi x/n}$.
The characters of $\mathbb{Z}_n$ *are the characters of* $\mathbb{Z}$ *which are* 1 *on* $n\mathbb{Z}$, *and so are of the form* $\xi(m) = \alpha^m$, *where* α *is the* n^{th} *root of* 1.
(5) *If* G_1, ..., G_n *are locally compact Abelian groups, then*

$$(G_1 \times \cdots \times G_n)\hat{} = \hat{G}_1 \times \cdots \times \hat{G}_n,$$

with $\xi = (\xi_1, ..., \xi_n) \in \hat{G}_1 \times \cdots \times \hat{G}_n$ *defined by* $\xi((x_1, ..., x_n)) = \prod_{j=1}^n \xi_j(x_j)$.
(6) $\widehat{\mathbb{R}^n} \simeq \mathbb{R}^n$, $\widehat{\mathbb{T}^n} \simeq \mathbb{Z}^n$, $\widehat{Z^n} \simeq \mathbb{T}^n$ *and* $\hat{G} = G$ *for any finite Abelian group.*

For $\xi, \eta \in \hat{G}$, let f and g satisfy $\xi(x) = e^{2\pi i f(x)}$ and $\eta(x) = e^{2\pi i g(x)}$. Then we see

$$\xi^{-1}(x) = e^{-2\pi i f(x)} = \overline{\xi(x)}$$

and

$$(\xi\eta)(x) = e^{2\pi i (f(x)+g(x)} = (\eta\xi)(x).$$

We then see that $\overline{\xi}$ and $\xi\eta$ are themselves characters, so $\hat{G}$ is a commutative multiplicative group (under pointwise multiplication), with the identity being the trivial character, $\xi_0(x) = 1$ identically. Now we want to give it a topology.

Definition 1.3.2. *A* ***basis*** *of a topology on a set* X *is a collection of opens sets* $\mathcal{B}$ *in* X *such that*
(1) *For each* $x \in X$, *there is some* $B \in \mathcal{B}$ *such that* $x \in B$.
(2) *If* $U \subseteq X$ *is open, then for each* $y \in U$, *there exists some* $B \in \mathcal{B}$ *such that* $x \in B$ *and* $B \subseteq U$.

Let $\mathcal{U}$ *be a family of neighborhoods of* $x \in X$. *We call* $\mathcal{U}$ *a* ***local basis*** *for* x *if for every neighborhood* V *of* x, *there exists some* $U \in \mathcal{U}$ *such that* $U \subseteq V$.

Every topology on a set X has a basis (though generally it is not unique). A collection of open sets $\mathcal{B}$ is a basis of the topology on X if and only if for every $x \in X$, the subcollection of $\mathcal{B}$ containing only neighborhoods of x is a local basis. In addition, if $Y \subseteq X$, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis on Y in the subspace topology. Likewise for a local basis of $x \in Y \subseteq X$.

Definition 1.3.3. *Given* $f \in A \subseteq C(G)$, $K \subseteq G$ *compact, and* $\varepsilon > 0$, *let*

$$B_K(f, \varepsilon) = \{g \in A : \sup\{|f(x) - g(x)| : x \in K\} < \varepsilon\}.$$

The sets $B_K(f, \varepsilon)$, *varying over choices of* K, f, *and* ε, *form a basis for a topology on* A. *This is call the* ***topology of compact convergence*** *or* ***topology of uniform convergence on compact sets***.

Since $\mathbb{C}$ is a metric space, $C(G)$, and therefore $\hat{G}$, is Hausdorff in the compact convergence topology [**Mun00**][Thm 46.8 and Exercise 6 in Section 46]. We can alternatively describe the topology as follows: ξ_n converges to ξ in $\hat{G}$ in the compact convergence topology if and only if for every compact $K \subseteq G$, ξ_n converges uniformly to $\xi \in \hat{G}$. It should be noted that one can show that the compact convergence topology on $\hat{G}$ can be shown to coincide with the *compact-open topology* [**Mun00**][Thm 46.8] and the weak* topology $\hat{G}$ inherits as a subspace of the dual space of $L^1(G)$ (see Chapters 2.3 and 3.3 in [**Fol15**]). We note that with the compact convergence topology, $\hat{G}$ is a closed subspace of $C(G)$ (basically, limits of homomorphisms are homomorphisms).

THEOREM 1.3.4 (Arscoli Theorem). *Suppose that X is a locally compact Hausdorff space and (Y, d) is a metric space. Then a subset of continuous functions from X to Y, $A \subseteq C(X, Y)$ has compact closure in the compact convergence topology if and only if*

- *For all $x \in X$, $\overline{\{\xi(x) : \xi \in A\}}$ is compact in Y.*
- *(Equicontinuity) For each $\varepsilon > 0$ and $x \in X$, there exists a neighborhood $U \subseteq X$ of x such that for all $y \in U$ and $\xi \in A$, $d(\xi(x), \xi(y)) < \varepsilon$.*

This can be found in [**Mun00**][Thm 47.1].

THEOREM 1.3.5. *If G is a locally compact Abelian group, then $\hat{G}$ is also a locally compact Abelian group, with the compact convergence topology.*

PROOF. As mentioned above, $\hat{G}$ is Hausdorff. Now we show that $\hat{G}$ is indeed a topological group. It is enough to show that for any two sequences $\{\xi_n\}_{n=1}^{\infty}$, $\{\eta_n\}_{n=1}^{\infty}$ of elements of $\hat{G}$ and any compact $K \subseteq G$ such that $\xi_n \to \xi$ and $\eta_n \to \eta$ uniformly on K, then $\xi_n \eta_n^{-1} \to \xi\eta^{-1}$ uniformly on K. This follows from the triangle inequality: for any $x \in K$,

$$\begin{aligned}
|\xi_n(x)\eta_n^{-1}(x) - \xi(x)\eta^{-1}(x)| &= |\xi_n(x)\eta_n^{-1}(x) - \xi(x)\eta_n^{-1}(x) + \xi(x)\eta_n^{-1}(x) - \xi(x)\eta^{-1}(x)| \\
&\leq |\xi_n(x)| \cdot |\eta^{-1}(x) - \eta_n^{-1}(x)| + |\xi_n(x) - \xi(x)| \cdot |\eta^{-1}(x)| \\
&= |\overline{\eta(x)} - \overline{\eta_n(x)}| + |\xi_n(x) - \xi(x)|.
\end{aligned}$$

Thus, if ξ_n and η_n converge uniformly on all compact subsets of G, so does $\xi_n\eta_n^{-1}$, giving us continuity of the product and inversion operations.

To show locally compactness, we need to show that each $\xi \in \hat{G}$ has a compact neighborhood. Since $\hat{G}$ is continuous under multiplication, we only need to show this for ξ_0.

For $1 > \varepsilon > 0$ and $K \subseteq G$ compact with positive Haar measure, let

$$N_{K,\varepsilon} := \{\xi \in \hat{G} : |\xi(x) - 1| < \varepsilon, \forall x \in K\}.$$

Since S^1 is compact, we know that for all $x \in G$, $\overline{\{\xi(x) : \xi \in N_{K,\varepsilon}\}}$ is compact, so we now only need to show equicontinuity of $N_{K,\varepsilon}$ at each point of G. However, since $\hat{G}$ is a set of homomorphisms from G to S^1,

$$|\xi(x) - \xi(0)| = |\xi(y)| \cdot |\xi(x) - \xi(0)| = |\xi(x + y) - \xi(y)|$$

for all $\xi \in \hat{g}$ and $x, y \in G$, so it suffices to show equicontinuity at $0 \in G$. We quickly see that for any $\xi \in N_{K,\varepsilon}$,

$$\sigma(K) \leq \int_K |\xi(x) - 1| d\sigma(x) + \left| \int_G \xi(x)\mathbb{1}_K(x) d\sigma(x) \right| \leq \sigma(K)\varepsilon + \left| \int_G \xi(x)\mathbb{1}_K(x) d\sigma(x) \right|,$$

so

$$\left| \int_G \xi(x)\mathbb{1}_K(x) d\sigma(x) \right| \geq (1 - \varepsilon)\sigma(K) > 0.$$

Lemma 1.2.6 and the continuity of inversion on G tells us that there must be some neighborhood $U \subseteq G$ of 0 such that for all $x \in G$ and $y \in U$,

$$\int_G |\mathbb{1}_K(x - y) - \mathbb{1}_K(x)| d\sigma(x) < \varepsilon,$$

so

$$\begin{aligned}
|\xi(y)-1| &\leq \frac{1}{(1-\varepsilon)\sigma(K)}\Big|(\xi(y)-1)\int_G \xi(x)\mathbb{1}_K(x)d\sigma(x)\Big| \\
&= \frac{1}{(1-\varepsilon)\sigma(K)}\Big|\int_G \xi(x+y)\mathbb{1}_K(x)d\sigma(x) - \int_G \xi(x)\mathbb{1}_K(x)d\sigma(x)\Big| \\
&= \frac{1}{(1-\varepsilon)\sigma(K)}\Big|\int_G \xi(x)\mathbb{1}_K(x-y) - \mathbb{1}_K(x)d\sigma(x)\Big| \\
&\leq \frac{1}{(1-\varepsilon)\sigma(K)}\int_G |\mathbb{1}_K(x-y) - \mathbb{1}_K(x)|d\sigma(x) \\
&< \frac{\varepsilon}{(1-\varepsilon)\sigma(K)}.
\end{aligned}$$

Theorem 1.3.4 (Arscoli Theorem) then tells us the closure of $N_{K,\varepsilon}$ in $C(G)$ is compact. Since $\hat{G}$ is a closed subspace of $C(G)$, and inherits the subspace topology, the closure of $N_{K,\varepsilon}$ in $\hat{G}$ is its closure in $C(G)$ and so is also compact in $\hat{G}$.

□

PROPOSITION 1.3.6. *If G is compact with normalized Haar measure ($\sigma(G) = 1$), then $\hat{G}$ is an orthonormal set in $L^2(G)$.*

PROOF. Let $\xi, \eta \in \hat{G}$. Then $\int_G |\xi(x)|^2 d\sigma(x) = 1$. If $\eta \neq \xi$, then there is some $x_0 \in G$ such that $\xi(x_0)\eta^{-1}(x_0) \neq 1$. We then have

$$\begin{aligned}
\int_G \xi(x)\overline{\eta(x)}d\sigma(x) &= \int_G \xi(x)\eta^{-1}(x)d\sigma(x) \\
&= \xi(x_0)\eta^{-1}(x_0)\int_G \xi(x-x_0)\eta^{-1}(x-x_0)d\sigma(x) \\
&= \xi(x_0)\eta^{-1}(x_0)\int_G \xi(x)\eta^{-1}(x)d\sigma(x)
\end{aligned}$$

where we have used the translation invariance of the Haar measure. We therefore must have $\int_G \xi(x)\overline{\eta(x)}d\sigma(x) = 0$. □

We note that if G is not compact, that this proof still tells us that $\hat{G}$ is an orthogonal set in $L^2(G)$

1.4. Fourier Transform

DEFINITION 1.4.1. *We define the **Fourier transform** $\mathcal{F} : L^1(G) \to C(\hat{G})$ by*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_G f(x)\overline{\xi(x)}d\sigma(x) \tag{1.4.1}$$

The definition of the Fourier transform does not require one to assume the image $L^1(G)$ is a set of continuous functions, however, the proof that this is indeed true uses Gelfand Theory of nonunital Banach algebras, and the details can be found in [**Fol15**][Chp 1.2] or in [**Rud90**][Appendix D4]. Here we show that the Fourier transform of an integrable function is sequentially continuous without this theory, and in most spaces of interest (when $\hat{G}$ is first-countable), this proves continuity.

PROOF OF SEQUENTIAL CONTINUITY. Suppose that $\xi_n \to \xi$ in the compact convergence topology on $\hat{G}$. Then for all $x \in G$, since $\{x\}$ is compact, $\xi_n(x) \to \xi(x)$, so ξ_n converges pointwise to ξ. Then for any $f \in L^1(G)$,

$$\begin{aligned}|\hat{f}(\xi_n) - \hat{f}(\xi)| &= \Big| \int_G f(x)(\overline{\xi_n(x)} - \overline{\xi(x)}) d\sigma(x)\Big| \\ &\leq \int_G |f(x)|\, |\xi_n(x) - \xi(x)| d\sigma(x).\end{aligned}$$

Since $|f(x)|\, |\xi_n(x) - \xi(x)| \leq 2|f(x)|$ for all $x \in G$ and $2|f| \in L^1(G)$, we have, by the Dominated Convergence Theorem, that $\lim_{n\to\infty} |\hat{f}(\xi_n) - \hat{f}(\xi)| = 0$, so $\hat{f}$ is sequentially continuous. □

LEMMA 1.4.2 (Riemann-Lebesgue lemma for locally compact Abelian groups). *For all $f \in L^1(G)$, $\hat{f} \in C_0(\hat{G})$.*

PROOF. Since $\hat{f}$ is continuous, we need only show that for every $\varepsilon > 0$, the set $K_{f,\varepsilon} := \{\xi \in \hat{G} : |\hat{f}(\xi)| \geq \varepsilon\}$ is compact. We may do this using Theorem 1.3.4 (Arscoli's Theorem). We immediately have that for all $x \in G$, $\overline{\{\xi(x) : \xi \in K_{f,\varepsilon}\}}$ is compact, since it is a closed subset of S^1. Since G is a locally compact Abelian group, we now need only show equicontinuity of $K_{f,\varepsilon}$ at $0 \in G$. The proof is quite similar to that of Theorem 1.3.5 at this point.

Let $\varepsilon > 0$, $\xi \in K_{f,\varepsilon}$, and $y \in G$. Then we have

$$\begin{aligned}|\xi(y) - 1|\varepsilon &\leq \Big|(\xi(y) - 1) \int_G f(x)\overline{\xi(x)} d\sigma(x)\Big| \\ &= \Big| \int_G f(x)\overline{\xi(x-y)} d\sigma(x) - \int_G f(x)\xi(x) d\sigma(x)\Big| \\ &= \Big| \int_G \big(f(x+y) - f(x)\big)\overline{\xi(x)} d\sigma(x)\Big| \\ &\leq \|L_y f - f\|_{L^1(G)},\end{aligned}$$

so

$$|\xi(y) - 1| \leq \frac{1}{\varepsilon}\|L_y f - f\|_{L^1(G)}.$$

Since L_y is continuous, by Lemma 1.2.6, we see that for every $\delta > 0$ there exists some neighborhood U of $0 \in G$ such that for all $y \in U$, $\|L_y f - f\|_{L^1(G)} < \varepsilon\delta$. Thus, on this neighborhood, $|\xi(y) - 1| < \delta$, giving us equicontinuity.

Theorem 1.3.4 (Arscoli Theorem) then tells us the closure of $K_{f,\varepsilon}$ in $C(G)$ is compact. Since $\hat{G}$ is a closed subspace of $C(G)$, and inherits the subspace topology, the closure of $K_{f,\varepsilon}$ in $\hat{G}$ is its closure in $C(G)$ and so is also compact in $\hat{G}$. □

THEOREM 1.4.3. *If G is compact, $\hat{G}$ is discrete. If G is discrete, $\hat{G}$ is compact.*

PROOF. Suppose that G is discrete. Then $\delta_0 : G \to \mathbb{C}$, with $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$, is in $L^1(G)$. Its Fourier Transform is given by

$$\hat{\delta_0}(\xi) = \int_G \delta_0(x)\overline{\xi(x)} d\sigma(x) = \overline{\xi(0)} = 1. \tag{1.4.2}$$

By Lemma 1.4.2 (Riemann Lebesgue),

$$\hat{G} = \{\xi \in \hat{G} : |\hat{\delta_0}(\xi)| \geq 1\}$$

is compact.

Now, suppose instead that G is compact. Then $1 \in L^2(G) \subset L^1(G)$, and by Proposition 1.3.6, we know that $\int_G \xi(x)d\sigma(x) = 0$ if ξ is not 1. So

$$\{\xi \in \hat{G} : |\hat{1}(\xi)| > \frac{1}{2}\} = \{\xi \in \hat{G} : \Big| \int_G \overline{\xi(x)} d\sigma(x) \Big| \geq \frac{1}{2}\} = \{1\},$$

meaning that the preimage of $\{x \in \mathbb{C} : |z| > \frac{1}{2}\}$ under continuous function $\hat{1}$, the singleton $\{1\}$ is open. Since $\hat{G}$ is a topological group, every singleton is open, giving us our claim. $\square$

THEOREM 1.4.4. *For all $a, b \in \mathbb{C}$, $f, g \in L^1(G)$, $y \in G$, and $\xi, \eta \in \hat{G}$, and defining $\tilde{f}(x) = f(-x)$, we have*

1. $\|\hat{f}\|_{L^\infty(\hat{G})} \leq \|f\|_{L^1(G)}$
2. $\mathcal{F}(af + bg) = a\hat{f} + b\hat{g}$
3. $\mathcal{F}(f * g) = \hat{f}\hat{g}$
4. $\overline{\mathcal{F}(f)} = \mathcal{F}(\,\overline{\tilde{f}}\,)$
5. $\mathcal{F}(L_y f)(\xi) = \xi(y)\hat{f}(\xi)$
6. $\mathcal{F}(\eta f)(\xi) = L_{\eta^{-1}}\hat{f}(\xi)$.

PROOF. Part 2 follows from the linearity of integration, and 1 can be seen by

$$\|\hat{f}\|_{L^\infty(\hat{G})} = \sup_{\xi \in \hat{G}} \Big| \int_G f(x)\overline{\xi(x)}d\sigma(x) \Big| \leq \sup_{\xi \in \hat{G}} \int_G |f(x)|\, |\overline{\xi(x)}| d\sigma(x) = \|f\|_{L^1(G)}.$$

For 3, we see, using Fubini's Theorem and the translation invariance of the Haar measure

$$\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \int_G (f * g)(x)\overline{\xi(x)}d\sigma(x) \\
&= \int_G \int_G f(x - y)g(y)\overline{\xi(x)}d\sigma(y)d\sigma(x) \\
&= \int_G g(y)\overline{\xi(y)} \int_G f(x - y)\overline{\xi(x - y)}d\sigma(x)d\sigma(y) \\
&= \int_G g(y)\overline{\xi(y)}d\sigma(y) \int_G f(x)\overline{\xi(x)}d\sigma(x).
\end{aligned}$$

For 4, we use the fact that σ is inverse-invariant:

$$\overline{\int_G f(x)\overline{\xi(x)}d\sigma(x)} = \int_G \overline{f(x)}\xi(x)d\sigma(x) = \int_G \overline{f(x)\xi(-x)}d\sigma(x) = \int_G \overline{f(-x)\xi(x)}d\sigma(x).$$

Again using the fact that $\overline{\xi(x)} = \xi^{-1}(x) = \xi(-x)$ for $\xi \in \hat{G}$ and $x \in G$, as well as translation-invariance of the Haar measure, we have

$$\int_G f(x + y)\overline{\xi(x)}d\sigma(x) = \int_G f(x)\overline{\xi(x - y)}d\sigma(x) = \overline{\xi(-y)} \int_G f(x)\overline{\xi(x)}d\sigma(x)$$

and

$$\int_G \eta(x)f(x)\overline{\xi(x)}d\sigma(x) = \int_G f(x)\overline{\xi(x)\overline{\eta(x)}}d\sigma(x) = \hat{f}(\xi\eta^{-1}),$$

giving us 5 and 6.

$\square$

THEOREM 1.4.5 (Stone-Weierstrass Theorem). *Let X be a locally compact Hausdorff space, and let A be a subalgebra of $C_0(X)$ such that*

(1) *(Separating) If $x \neq y$ $(x, y \in X)$, $f(x) \neq f(y)$ for some $f \in A$,*
(2) *If $x \in X$, $f(x) \neq 0$ for some $f \in A$,*
(3) *If $f \in A$, $\overline{f} \in A$.*

Then A is dense in $C_0(X)$.

This can be found as a special case of Bishop's Theorem in [**Rud91**][Thm 5.7] (see the discussion below the statement of Bishop's Theorem).

COROLLARY 1.4.6. *The space $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$*

PROOF. We already know that $\mathcal{F}(L^1(G))$ is closed under conjugation, by part 4 of Theorem 1.4.4.

Now, let $\xi_1, \xi_2 \in \hat{G}$, with $\xi_1 \neq \xi_2$. There must be some $y \in G$ for which $\xi_1(y) - \xi_2(y) = z \neq 0$. Since ξ_1 and ξ_2 are continuous there is some open set $U = \{x \in G : |z - (\xi_1(x) - \xi_2(y))| < \frac{|z|}{2}\}$. Since G is locally compact, there exists some neighborhood A of y with compact closure. Since $y \in A \cap U$, we know $A \cap U$ is non-empty and open and has compact closure ($\overline{A \cap U} \subset \overline{A}$), and so has finite positive Haar measure. Let $f = \mathbb{1}_{A \cap U}$. Then

$$\begin{aligned}\hat{f}(\xi_1) - \hat{f}(\xi_2) &= \int_{A\cap U} \overline{\xi_1(x)} - \overline{\xi_2(x)} d\sigma(x)\\ &= \int_{A\cap U} \overline{z} - \overline{z} + \overline{\xi_1(x)} - \overline{\xi_2(x)} d\sigma(x)\\ &= \sigma(A\cap U)\overline{z} - \overline{\int_{A\cap U} z - (\xi_1(x) - \xi_2(x)) d\sigma(x)}\end{aligned}$$

and since

$$\Big| \int_{A\cap U} z - (\xi_1(x) - \xi_2(x)) d\sigma(x) \Big| \leq \int_{A\cap U} |z - (\xi_1(x) - \xi_2(x))| d\sigma(x) \leq \sigma(A\cap U)\frac{|z|}{2},$$

$\hat{f}(\xi_1) \neq \hat{f}(\xi_2)$.

Likewise, let $V = \{x \in G : |1 - \xi_1(x)| < \frac{1}{2}\}$ and J be a neighborhood of 0 in G with compact closure, so $V \cap J$ has finite positive Haar measure. Then, with $f = \mathbb{1}_{J\cap V}$,

$$\hat{f}(\xi_1) = \int_{V\cap J} \overline{\xi_1(x)} d\sigma(x) = \sigma(V\cap J) - \int_{V\cap J} 1 - \overline{\xi_1(x)} d\sigma(x) \neq 0.$$

Theorem 1.4.5 (Stone-Weierstrass Theorem) finishes our proof.

□

PROPOSITION 1.4.7. *Let $g : G \times \hat{G} \to S^1$ be defined by $g(x, \xi) = \xi(x)$. Then g is continuous.*

PROOF. Let $x \in G$, $\xi \in \hat{G}$, $f \in L^1(G)$, and $\varepsilon > 0$. Due to the continuity of L_z from Lemma 1.2.6 and $\mathcal{F}(L_x f)$ from Lemma 1.4.2, there exists neighborhoods V of x and U of ξ such that for all $y \in V$ and $\eta \in U$,

$$\|L_x f - L_y f\|_{L^1(G)} < \varepsilon \text{ and } |\mathcal{F}(L_x f)(\xi) - \mathcal{F}(L_x f)(\eta)| < \varepsilon. \tag{1.4.3}$$

We therefore have, by the Triangle inequality and Theorem 1.4.4

$$\begin{aligned}|\mathcal{F}(L_x f)(\xi) - \mathcal{F}(L_y f)(\eta)| &\leq |\mathcal{F}(L_x f)(\xi) - \mathcal{F}(L_x f)(\eta)| + |\mathcal{F}(L_x f)(\eta) - \mathcal{F}(L_y f)(\eta)|\\ &< \varepsilon + \|L_x f - L_y f\|_{L^1(G)} < 2\varepsilon.\end{aligned}$$

Thus, $h_f : G \times \hat{G} \to S^1$ defined by $h(x,\xi) = \xi(x)\hat{f}(\xi) = \mathcal{F}(L_x f)(\xi)$ is continuous. This immediately means that $g(x,\xi) = \xi(x)$ is continuous in x. Now, let $p \in C_0(\hat{G})$ such that $p(\eta) > 0$ for all $\eta \in \hat{G}$. Since $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$ by Corollary 1.4.6, there exists some sequence of functions $p_n \in L^1(G)$ such that $\hat{p}_n \to p$ in $C_0(\hat{G})$. Thus, for n sufficiently large, we have for any $x \in G$ and $\eta \in \hat{G}$

$$\begin{aligned} |p(\xi)\xi(x) - p(\eta)\eta(x)| &\le |p(\xi)\xi(x) - \hat{f}_n(\xi)\xi(x)| + |\hat{f}_n(\xi)\xi(x) - \hat{f}_n(\eta)\eta(x)| + |\hat{f}_n(\eta)\eta(x) - p(\eta)\eta(x)| \\ &= |p(\xi) - \hat{f}_n(\xi)| + |\hat{f}_n(\xi)\xi(x) - \hat{f}_n(\eta)\eta(x)| + |\hat{f}_n(\eta) - p(\eta)| \\ &\le 2\|p - \hat{f}_n\|_{L^\infty(\hat{G})} + |\mathcal{F}(L_x f_n)(\xi) - \mathcal{F}(L_x f_n)(\eta)| \\ &< 2\varepsilon + |\mathcal{F}(L_x f_n)(\xi) - \mathcal{F}(L_x f_n)(\eta)|. \end{aligned}$$

By the discussion above, we know we may then choose a neighborhood V_n of ξ such for $\eta \in V_n$, $|p(\xi)\xi(x) - p(\eta)\eta(x)| < 3\varepsilon$. Thus $p(\xi)\xi(x)$ is continuous as a function of ξ and x. Since $p(\xi) > 0$, we then have $g(x,\xi) = \frac{p(\xi)\xi(x)}{p(\xi)}$ for all $x \in G$ and $\xi \in \hat{G}$, and so g is indeed continuous in both variables. ☐

1.5. Bochner's Theorem and Fourier Inversion

1.5.1. Fourier-Stieltjes transform.

Definition 1.5.1. *For a complex Radon measure μ on G, we define the **total variation** of μ by*

$$|\mu|(E) = \sup_P \sum_{A \in P} |\mu(A)|, \quad E \text{ a Borel subset of } G,$$

where the supremum ranges over all countable, disjoint partitions P of E into Borel sets. The set of complex Radon measures μ with $|\mu|(G) < \infty$ will be denoted by $M(G)$.

Note that the total variation $|\mu|$ is a positive Radon measure with $|\mu(E)| \le |\mu|(E)$ for all Borel sets $E \subseteq G$.

The Fourier Transform can be extended $M(G)$, which contains $L^1(G)$ as for all $f \in L^1(G)$, $f(x)d\sigma(x)$ defines a complex Radon measure with bounded total variation. For any $\mu \in M(G)$ the **Fourier-Stieltjes transform** of μ by

$$\mathcal{F}(\mu)(\xi) = \hat{\mu}(\xi) := \int_G \overline{\xi(x)}\,d\mu(x), \quad \xi \in \hat{G}.$$

Proposition 1.5.2. *For all $\mu \in M(G)$, $\hat{\mu}$ is a bounded continuous function on $\hat{G}$.*

Proof. Let $\varepsilon > 0$, $K \subseteq G$ compact with $|\mu|(G \backslash K) < \varepsilon$, $\xi \in \hat{G}$, and $N_{K,\varepsilon}(\xi) := \{\eta \in \hat{G} : |\eta(x) - \xi(x)| < \varepsilon,\ \forall x \in K\}$. Then for any $\eta \in N_{K,\varepsilon}(\xi)$,

$$\begin{aligned} |\hat{\mu}(\xi) - \hat{\mu}(\eta)| &\le \int_G |\xi(x) - \eta(x)|\,d|\mu|(x) \\ &\le 2|\mu|(G \setminus K) + |\mu|(K) \sup_{x \in K} |\xi(x) - \eta(x)| \\ &\le (2 + |\mu|(G))\varepsilon, \end{aligned}$$

giving us continuity. For boundedness, we have, for all $\xi \in \hat{G}$,

$$|\hat{\mu}(\xi)| \le \int_G |\xi(x)|\,d|\mu|(x) \le |\mu|(G) < \infty.$$

☐

For any $\nu \in M(\hat{G})$, we will define the Fourier-Stieltjes transform of ν by

$$\mathcal{F}(\nu)(x) = \phi_\nu(x) := \int_{\hat{G}} \xi(x) d\nu(\xi), \quad x \in G. \tag{1.5.1}$$

PROPOSITION 1.5.3. *The Fourier-Stieltjes transform on $M(\hat{G})$ is an injective map to the space of bounded continuous functions on G, and*

$$\|\phi_\nu\|_{L^\infty(G)} \leq |\nu|(\hat{G}) \tag{1.5.2}$$

PROOF. The proof that the Fourier-Stieltjes transform maps to the space of bounded continuous functions in G and is norm-decreasing is similar to that in Proposition 1.5.2. For injectivity, suppose that $\phi_\nu = 0$ and $f \in L^1(G)$. Then, by Fubini's Theorem

$$0 = \int_G f(x)\phi_\nu(x) d\sigma(x) = \int_G \int_{\hat{G}} f(x)\xi(x) d\nu(\xi) d\sigma(x) = \int_{\hat{G}} \hat{f}(\xi^{-1}) d\nu(\xi).$$

By Corollary 1.4.6, $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$, so $\mu = 0$. □

1.5.2. Functions of Positive type and Positive definite functions. For all $f \in L^1(G)$, let us define $f^*(x) := \overline{f(-x)}$. We call $\phi \in L^\infty(G)$ to be a **function of positive type** if

$$\int_G (f^* * f)(x)\phi(x) d\sigma(x) = \int_G \int_G f(x)\overline{f(y)}\phi(x-y) d\sigma(x) d\sigma(y) \geq 0 \quad \forall f \in L^1(G).$$

We see these definitions are equivalent, as

$$\begin{aligned}
\int_G (f^* * f)(x)\phi(x) d\sigma(x) &= \int_G \int_G f(x-y)\overline{f(-y)}\phi(x) d\sigma(y) d\sigma(x) \\
&= \int_G \int_G f(x+y)\overline{f(y)}\phi(x) d\sigma(y) d\sigma(x) \\
&= \int_G \int_G f(x)\overline{f(y)}\phi(x-y) d\sigma(y) d\sigma(x).
\end{aligned}$$

DEFINITION 1.5.4. *We call $\phi \in L^\infty(G)$ **positive definite** if, for all $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{C}$, and $x_1, ..., x_N \in G$,*

$$\sum_{i,j=1}^N c_i \overline{c_j} \phi(x_i - x_j) \geq 0.$$

In other words, the matrix $[\phi(x_i - x_j)]_{i,j=1}^N$ is positive semi-definite for all $N \in \mathbb{N}$ and $x_1, ..., x_N \in G$.

We see that for $n = 2$, $x_1 = 0$, this definition means that the matrix

$$\begin{pmatrix} \phi(0) & \phi(x_2) \\ \phi(-x_2) & \phi(0) \end{pmatrix}$$

is positive semi-definite, so $\phi(-x_2) = \overline{\phi(x_2)}$ and $\phi(0)^2 - \phi(x_2)\phi(-x_2) \geq 0$. Thus $|\phi(x)| \leq \phi(0)$ for all $x \in G$. Thus, positive definite functions are bounded.

LEMMA 1.5.5. *Every character $\xi \in \hat{G}$ is of positive type.*

PROOF. Let $f \in L^1(G)$. Then

$$\int_G \int_G f(x)\overline{f(y)}\xi(x-y) d\sigma(x) d\sigma(y) = \Big(\int_G f(x)\xi(x) d\sigma(x)\Big)\overline{\Big(\int_G f(y)\xi(y) d\sigma(y)\Big)} = |\hat{f}(\xi^{-1})| \geq 0.$$

□

LEMMA 1.5.6. *If f, g are functions of positive type, then so are $f+g$ and fg.*

PROOF. Let $N \in \mathbb{N}$ and $x_1, ..., x_N \in G$. Since $[f(x_i - x_j)]_{i,j=1}^N$ and $[g(x_i - x_j)]_{i,j=1}^N$ are positive semi-definite, so we immediately have that their sum $[(f+g)(x_i - x_j)]_{i,j=1}^N$ is as well. The Schur product theorem tells us that the Hadamard product of these two matrices, $[f(x_i - x_j)g(x_i - x_j)]_{i,j=1}^N$, is positive semi-definite. □

LEMMA 1.5.7. *If ϕ is continuous and bounded on G, then the following are equivalent*

(1) *ϕ is of positive type.*
(2) *ϕ is positive definite.*
(3) *$\int_G (f^* * f)(x)\phi(x)d\sigma(x) \geq 0$ for all $f \in C_c(G)$.*

This is Proposition 3.35 in [**Fol15**], and a complet proof is given there. Here, we will just prove that positive type implies positive definiteness.

PROOF. Let ψ_U be an approximate identity, and for $c_1, ...c_n \in \mathbb{C}$ and $x_1, ..., x_n \in G$, let $f_U := \sum_{j=1}^n c_j L_{x_j}\psi_U$. Then

$$0 \leq \int_G \int_G f_U(z)\overline{f_U(y)}\phi(z-y)d\sigma(z)d\sigma(y) \quad = \sum_{j,k=1}^n c_j\overline{c_k}\int_G\int_G \psi_U(z+x_j)\overline{\psi_U(y+x_k)}\phi(z-y)d\sigma(z)d\sigma(y).$$

Since ϕ is continuous,

$$\lim_{U\to\{0\}} \sum_{j,k=1}^n c_j\overline{c_k}\int_G\int_G \psi_U(z+x_j)\overline{\psi_U(y+x_k)}\phi(z-y)d\sigma(z)d\sigma(y) = \sum_{j,k=1}^N c_j\overline{c_k}\phi(x_j - x_k)$$

so the sum on the right is nonnegative. □

LEMMA 1.5.8. *If $f \in L^2(G)$, $f^* * f \in \mathcal{P}(G)$.*

PROOF. For any $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{C}$ and $x_1, ..., x_n \in G$, we have that

$$\sum_{i,j=1}^n c_i\overline{c_j}(f^* * f)(x_i - x_j) = \int_G \sum_{i,j=1}^n c_i\overline{c_j}\overline{f(-y)}f(x_i - x_j - y)d\sigma(y) \tag{1.5.3}$$

$$= \int_G \sum_{i,j=1}^n c_i\overline{c_j}\overline{f(x_j - y)}f(x_i - y)d\sigma(y) \tag{1.5.4}$$

$$= \int_G \overline{\Big(\sum_{i=1}^n c_i f(x_i - y)\Big)}\Big(\sum_{j=1}^n c_j f(x_j - y)\Big)d\sigma(y) \tag{1.5.5}$$

$$= \int_G \Big|\sum_{i=1}^n c_i f(x_i - y)\Big|^2 d\sigma(y) > 0. \tag{1.5.6}$$

By Lemma 1.2.7, $f^* * f \in C_0$ and is bounded, and therefore is positive definite. Lemma 1.5.7 finishes our proof. □

1.5.3. Bochner's Theorem.

1.5.3.1. *Required Functional Analysis and Gelfand Theory.*

DEFINITION 1.5.9. *We call a linear map $T : X \to Y$ between normed vector spaces X and Y (with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$) a* ***linear operator****. We define the* ***operator norm*** *on the space of linear operators from X to Y by*

$$\|T\|_{X,Y,op} = \sup\{\|T(x)\|_Y : x \in X, \|x\| \leq 1\}, \tag{1.5.7}$$

and say that call T a ***bounded linear operator*** *if $\|T\|_{X,Y,op} < \infty$, i.e., there is some $c \geq 0$ such that for all $x \in X$,*

$$\|T(x)\|_Y \leq c\|x\|_X. \tag{1.5.8}$$

DEFINITION 1.5.10. *For a normed vector space X, we call a linear operator $T : X \to \mathbb{C}(\mathbb{R})$ a* ***linear functional*** *(or functional for short). If X is an ordered vector space $(X, \geq)$, then we T is* ***positive*** *if for all $v \in X$ with $v \geq 0$,*

$$T(v) \geq 0. \tag{1.5.9}$$

THEOREM 1.5.11. *Let X and Y be normed vector spaces, and $T : X \to Y$ be a linear operator. Then T is continuous if and only if it is bounded.*

This can be found in [**Kre78**][Thm 2.7-9].

THEOREM 1.5.12 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space and A be a positive linear functional on $C_c(X)$. Then there is a unique positive Radon measure μ, such that*

$$A(f) = \int_X f(x)d\mu(x) \quad \forall f \in C_c(X).$$

In this case, the operator norm on A is the total variation of μ and A is a positive operator if and only if μ is a positive measure.

This can be found in [**Rud87**][Thm 2.14].

THEOREM 1.5.13 (Riesz-Markov Representation Theorem). *Let X be a locally compact Hausdorff space. For any continuous linear functional A on $C_0(X)$, there is a unique complex Radon measure with bounded total variation on X, $\mu \in M(X)$, such that*

$$A(f) = \int_X f(x)d\mu(x) \quad \forall f \in C_0(X).$$

This can be found in [**Rud87**][Thm 6.19].

PROPOSITION 1.5.14 (Spectral Radius Formula). *Let $f_1 \in L^1(G)$, and for $n \in \mathbb{N}$ define $f_{n+1} = f_1 * f_n$. Then*

$$\lim_{n\to\infty} \|f_n\|_{L^1(G)}^{1/n} = \|\hat{f}_1\|_{L^\infty(\hat{G})}. \tag{1.5.10}$$

The upper bound follows from Theorem 1.4.4 and Proposition 1.2.9, which give us

$$\begin{aligned}
\|\hat{f}_1\|^n_{L^\infty(\hat{G})} &= \|\hat{f}_1^n\|_{L^\infty(\hat{G})} \\
&= \|\hat{f}_n\|_{L^\infty(\hat{G})} \\
&\leq \|f_n\|_{L^1(G)}.
\end{aligned}$$

The lower bound follows from a combination of showing that series of the form $\sum \frac{\phi(f_n)}{\lambda^n}$ (for ϕ a bounded linear operator) or $\sum \frac{f_n}{\lambda^n}$ are analytic (in other words a complex derivative exists) as a function of λ, using this to show boundedness of terms, and then using the Banach-Steinhaus (Uniform Boundedness Principle) to finish the proof (or at least that is one way).

See Chapter 10 (particularly Theorem 10.13) in [**Rud91**], Appendix D6 in [**Rud90**], or Chapter 1 (particularly Theorems 1.8, 1.13, and 1.30) in [**Fol15**] for more of an introduction.

1.5.3.2. *Bochner's Theorem.*

LEMMA 1.5.15. *If $\nu \in M(\hat{G})$ is a positive measure, then ϕ_ν is a continuous function of positive type.*

PROOF. We have continuity and boundedness by Proposition 1.5.3. For all $f \in L^1(G)$,

$$\begin{aligned}\int_G \int_G f(x)\overline{f(y)}\phi_\nu(x-y)d\sigma(x)d\sigma(y) &= \int_G \int_G \int_{\hat{G}} f(x)\overline{f(y)}\xi(x-y)d\nu(\xi)d\sigma(x)d\sigma(y)\\ &= \int_{\hat{G}} |\hat{f}(\xi)|^2 d\nu(\xi) \geq 0.\end{aligned}$$

□

THEOREM 1.5.16 (Bochner's Theorem). *A continuous function ϕ on G is of positive type if and only if there is a (unique) positive measure $\nu \in M(\hat{G})$ such that $\phi(x) = \phi_\nu(x)$.*

PROOF. We proved one direction in Lemma 1.5.15, and uniqueness follows from injectivity from Proposition 1.5.3.

Without loss of generality, assume ϕ is of positive type, with $\phi(0) = 1$, and define the linear functional

$$T_\phi(f) = \int_G f(x)\phi(x)d\sigma(x) \quad f \in L^1(G).$$

We can quickly see that $|T_\phi(f)| \leq \|f\|_{L^1(G)}$, so this functional is bounded.

Let ψ_U be an approximate identity. Then, by Theorem 1.2.12, $\psi_U^* * f \to f$ in $L^1(G)$, so

$$\lim_{U\to\{0\}} T_\phi\Big(\psi_U^* * f\Big) = \lim_{U\to\{0\}} \int_G \phi(x)(\psi_U^* * f)(x)d\sigma(x) = \int_G \phi(x)f(x)d\sigma(x) = T_\phi(f).$$

If $\mathrm{supp}(\psi_U) \subseteq U$, then $\mathrm{supp}(\psi_U^* * \psi_U) \subseteq U - U$, and

$$\int_G (\psi_U^* * \psi_U)(x)d\sigma(x) = |\int_G \psi_U(x)d\sigma(x)|^2 = 1,$$

by Fubini's Theorem, so $\psi_U^* * \psi_U$ is an approximate identity and

$$\lim_{U\to\{0\}} \int_G \phi(x)(\psi_U^* * \psi_U)(x)d\sigma(x) = \phi(0) = 1.$$

Now we consider for all $f, g \in L^1(G)$

$$\langle f, g\rangle_\phi := T_\phi(f * g^*) = \int_G \int_G f(x)\overline{g(y)}\phi(x-y)d\sigma(x)d\sigma(y),$$

which is linear in f, and satisfies $\langle f, g\rangle_\phi = \overline{\langle g, f\rangle_\phi}$ and $\langle f, f\rangle_\phi \geq 0$, due to ϕ being of positive type. Thus, this is a positive Hermitian form, and we may apply the Cauchy-Schwarz inequality to find

$$\Big|\int_G \phi(x)(\psi_U^* * f)(x)d\sigma(x)\Big|^2 \leq \int_G \phi(x)(f^* * f)(x)d\sigma(x)\int_G \phi(y)(\psi_U^* * \psi_U)(y)d\sigma(y), \tag{1.5.11}$$

so

$$\Big|\int_G \phi(x)f(x)d\sigma(x)\Big|^2 \leq \int_G \phi(x)(f^* * f)(x)d\sigma(x). \tag{1.5.12}$$

Let $h_0 = f^* * f$ and for $n \in \mathbb{N}_0$, $h_{n+1} = h_n * h_n = h_n^* * h_n$. Since $\|\phi\|_{L^\infty(G)} = \phi(0) = 1$, (1.5.12) gives us

$$\Big|\int_G \phi(x)f(x)d\sigma(x)\Big| \leq \Big|\int_G \phi(x)h_0(x)d\sigma(x)\Big|^{\frac{1}{2}} \leq \cdots \leq \Big|\int_G \phi(x)h_n(x)d\sigma(x)\Big|^{\frac{1}{2^{n+1}}} \leq \|h_n\|_{L^1(G)}^{2^{-n-1}}.$$

Applying Proposition 1.5.14 (Spectral Radius Formula) we see that

$$|T_\phi(f)| \leq \lim_{n\to\infty} \|h_n\|_{L^1(G)}^{2^{-n-1}} = \|\hat{h}\|_{L^\infty(\hat{G})}^{\frac{1}{2}} = \||\hat{f}|^2\|_{L^\infty(\hat{G})}^{\frac{1}{2}} = \|\hat{f}\|_{L^\infty(\hat{G})}.$$

Thus, if $f, g \in L^1(G)$ with $\hat{g} = \hat{f}$, then $T_\phi(g) = T_\phi(f)$. Thus, we may define a bounded linear functional S_ϕ on $\mathcal{F}(L^1(G))$ such that for all $f \in L^1(G)$, $S_\phi(\hat{f}) = T_\phi(f)$, with operator norm at most one. Since $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$, it extends to a linear functional on $C_0(\hat{G})$ of norm at most 1 (this can be done via an approximation argument). Since S_ϕ is bounded and linear, it is continuous. By Theorem 1.5.13 (Riesz-Markov Representation Theorem), there is some $\nu \in M(\hat{G})$ with $|\tilde{\nu}|(\hat{G}) \leq 1$ such that

$$\int_G \phi(x)f(x)d\sigma(x) = S_\phi(\hat{f}) = \int_{\hat{G}} \hat{f}(\xi)d\tilde{\nu}(\xi) = \int_G\int_{\hat{G}} f(x)\overline{\xi(x)}d\tilde{\nu}(\xi)d\sigma(x).$$

This means that $\phi(x) = \int_{\hat{G}} \xi(x)d\nu(\xi)$, where $d\nu(\xi) = d\tilde{\nu}(\xi^{-1})$. Since

$$1 = \phi(0) = \nu(\hat{G}) \leq |\nu|(\hat{G}) = 1,$$

$\nu(\hat{G}) = |\nu|(\hat{G})$ and so ν is a positive measure. □

1.5.4. Fourier Inversion. Let $\mathcal{B}(G) := \{\phi_\mu : \mu \in M(\hat{G})\}$, with ϕ_μ as in (1.5.1), and $\mathcal{P}(G)$ be the continuous functions of positive type. Bochner's Theorem tells us the $\mathcal{B}(G)$ is the linear span of $\mathcal{P}(G)$. In addition, we see the map from $M(\hat{G})$ to $\mathcal{B}(G)$ is a bijection: surjectivity is immediate, and if $\phi_\mu = 0$, then for all $f \in L^1(G)$,

$$0 = \int_G f(x)\phi_\mu(x)d\sigma(x) = \int_G\int_{\hat{g}} f(x)\xi(x)d\mu(\xi)d\sigma(x) = \int_{\hat{G}} \hat{f}(\xi^{-1})d\mu(\xi),$$

and since $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$, this means that $\int_{\hat{G}} g(\xi)d\mu(\xi) = 0$ for all $g \in C_0$, so μ must be the zero measure, giving us injectivity. We denote the inverse map of $\mu \mapsto \phi_\mu$ by $\phi \mapsto \mu_\phi$, so $\phi_{\mu_\phi} = \phi$.

LEMMA 1.5.17. *If $f, g \in \mathcal{B}(G) \cap L^1(G)$, then $\hat{f}d\mu_g = \hat{g}d\mu_f$.*

PROOF. Let $h \in L^1(G)$. Then by Fubini's Theorem

$$\begin{aligned}\int_{\hat{G}} \hat{h}(\xi)\hat{g}(\xi)d\mu_f(\xi) &= \int_{\hat{G}}\int_G (h*g)(x)\xi(x)d\sigma(x)d\mu_f(\xi)\\ &= \int_G (h*g)(x)f(-x)d\sigma(x)\\ &= \big((h*g)*f\big)(0)\\ &= \big((h*f)*g\big)(0)\\ &= \int_{\hat{G}} \hat{h}(\xi)\hat{f}(\xi)d\mu_g(\xi).\end{aligned}$$

The density of $\mathcal{F}(L^1(G))$ in $C_0(\hat{G})$ finishes the proof. □

LEMMA 1.5.18. *If $K \subseteq \hat{G}$ is compact, there exists $f \in C_c(G) \cap \mathcal{P}(G)$ such that $\hat{f} \geq 0$ on $\hat{G}$ and $\hat{f} > 0$ on K.*

Proof. Let $h \in C_c(G)$ such that $\hat{h}(1) = \int_G h(x)d\sigma(x) = 1$, and let $g = h^* * h$. Then, by Theorem 1.4.4 $\hat{g}(\xi) = |\hat{h}(\xi)|^2$, so $\hat{g} \geq 0$ and $\hat{g}(1) = 1$, so there is a neighborhood U of 1 in $\hat{G}$ such that $\hat{g} > 0$, since $\hat{g}$ is continuous. Defining $kU = \{k\xi : \xi \in U\}$ for $k \in K$, we see by compactness that, for some $n \in \mathbb{N}$, there is some $k_1, ..., k_n \in K$ such that $K \subset \cup_{j=1}^n k_j U$. Let $f(x) := (\sum_{j=1}^n k_j(x))g(x)$. Then by Theorem 1.4.4,

$$\hat{f}(\xi) = \sum_{j=1}^n \hat{g}(\xi k_j^{-1}),$$

so $\hat{f} > 0$ on K and $\hat{f} \geq 0$ on $\hat{G}$. If $H = \operatorname{supp}(h)$, then $\operatorname{supp}(f) = \operatorname{supp}(g) = H - H = \{h_1 - h_2 : h_1, h_2 \in H\}$, which is compact, so $f \in C_c(G)$. Finally, we see that for all $b \in L^1(G)$

$$\begin{aligned}
\int_G (b^* * b)(x)f(x)d\sigma(x) &= \sum_{j=1}^n \int_G \int_G \overline{b(-y)}b(x-y)k_j(x)g(x)d\sigma(y)d\sigma(x) \\
&= \sum_{j=1}^n \int_G \int_G \overline{b(-y)}b(x-y)\overline{k_j(-y)}k_j(x-y)g(x)d\sigma(y)d\sigma(x) \\
&= \sum_{j=1}^n \int_G \big((k_j b)^* * (k_j b)\big)g(x)d\sigma(x) \geq 0,
\end{aligned}$$

as g is of positive type, due to Lemma 1.5.8. $\square$

Theorem 1.5.19 (Fourier Inversion Theorem 1). *Suppose that the dual Haar measure $\sigma_{\hat{G}}$ is suitably normalized with respect to the Haar measure on G. If $f \in \mathcal{B}(G) \cap L^1(G)$, then $\hat{f} \in L^1(G)$ and*

$$f(x) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)d\sigma_{\hat{G}}(\xi), \quad \forall x \in G. \tag{1.5.13}$$

Proof. Let $h \in C_c(\hat{G})$, with $K = \operatorname{supp}(h)$. By Lemma 1.5.18, there is some $f \in L^1(G) \cap \mathcal{P}(G)$ such that $\hat{f} > 0$ on $\operatorname{supp}(h)$ and $\hat{f} \geq 0$ on $\hat{G}$. Let

$$I(h) = \int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\xi)} d\mu_f(\xi).$$

If g is another such function, then by Lemma 1.5.17, we have

$$\int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\xi)} d\mu_f(\xi) = \int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\xi)\hat{g}(\xi)} \hat{g}(\xi) d\mu_f(\xi) = \int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\xi)\hat{g}(\xi)} \hat{f}(\xi) d\mu_g(\xi) = \int_{\hat{G}} \frac{h(\xi)}{\hat{g}(\xi)} d\mu_g(\xi),$$

so $I(h)$ does not depend on our choice of f, and since μ_f is a positive measure (by Bochner's Theorem) and $\hat{f} \geq 0$, $I(h) \geq 0$ if $h \geq 0$. Now, suppose that $m \in C_c(\hat{G})$ with $J = \operatorname{supp}(m)$. Then, again by Lemma 1.5.18, there is some $f \in\in L^1(G) \cap \mathcal{P}(G)$ such that $\hat{f} > 0$ on $K \cup J$ (which is compact) and $\hat{f} \geq 0$ on $\hat{G}$. Due to $I(h)$ dependence only on h, we then have, for $a, b \in \mathbb{C}$,

$$I(ah + bm) = \int_{\hat{G}} \frac{ah(\xi) + bm(\xi)}{\hat{f}(\xi)} d\mu_f(\xi) = a \int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\xi)} d\mu_f(\xi) + b \int_{\hat{G}} \frac{m(\xi)}{\hat{f}(\xi)} d\mu_f(\xi) = aI(h) + bI(m), \tag{1.5.14}$$

so I is a positive linear functional on $C_c(\hat{G})$.

Moreover, if $p \in \mathcal{B}(G) \cap L^1(G)$, then $\operatorname{supp}(\hat{p}h) \subseteq \operatorname{supp}(h)$ is compact, so again by Lemma 1.5.17,

$$I(\hat{p}h) = \int_{\hat{G}} \frac{\hat{p}(\xi)h(\xi)}{\hat{f}(\xi)} d\mu_f(\xi) = \int_{\hat{G}} h(\xi) d\mu_g(\xi).$$

Appropriate choices of p and h (such as p so that μ_p is a positive on $\text{supp}(h)$ and $h \geq 0$) give us $I \neq 0$, so I is non-trivial.

Now, let $\eta \in \hat{G}$. Then for all $x \in G$,

$$\int_{\hat{G}} \xi(x) d\mu_f(\eta\xi) = \int_{\hat{G}} \eta^{-1}(x)\eta(x)\xi(x) d\mu_f(\eta\xi) = \overline{\eta(x)} f(x),$$

so $d\mu_f(\eta\xi) = d\mu_{\overline{\eta} f}(\xi)$. Let $f \in L^1(G) \cap \mathcal{P}(G)$ such that $\hat{f} > 0$ on $K \cup \text{supp}(L_{\eta^{-1}} h)$ (such a function exists, but 1.5.18. By Lemmas 1.5.5 and 1.5.6 (Schur's Lemma) and the fact that $|\overline{\eta} f| = |f|$, we see that $\overline{\eta} f \in L^1(G) \cap \mathcal{P}(G)$, and since, by Theorem 1.4.4, $\mathcal{F}(\overline{\eta} f)(\xi) = \hat{f}(\eta\xi)$, we see that $\mathcal{F}(\overline{\eta} f)(\xi) > 0$ on K. Thus, we have

$$I(L_{\eta^{-1}} h) = \int_{\hat{G}} \frac{h(\eta^{-1}\xi)}{\hat{f}(\xi)} d\mu_f(\xi) = \int_{\hat{G}} \frac{h(\xi)}{\hat{f}(\eta\xi)} d\mu_f(\eta\xi) = \int_{\hat{G}} \frac{h(\xi)}{\mathcal{F}(\overline{\eta} f)(\xi)} d\mu_{\overline{\eta} f}(\xi) = I(h),$$

so I is translation invariant.

By Theorem 1.5.12 (Riesz Representation Theorem), there is a unique positive Radon measure μ such that $I(h) = \int_{\hat{G}} h(\xi) d\mu(\xi)$ for all $h \in C_c(\hat{G})$. Since μ must be translation invariant, by Theorem 1.2.3 (Haar's Theorem), this must be a Haar measure on $\hat{G}$ (with the correct normalization).

By (1.5.14), we know that for $f \in L^1(G) \cap \mathcal{B}(G)$ and $h \in C_c(\hat{G})$,

$$\int_{\hat{G}} h(\xi)\hat{f}(\xi) d\sigma_{\hat{G}}(\xi) = I(\hat{f} h) = \int_{\hat{G}} h(\xi) d\mu_f(\xi),$$

so $\hat{f}(\xi) d\sigma_{\hat{G}}(\xi) = d\mu_f(\xi)$. Since μ_f has bounded total variation (in other words, the positive and negative (real and imaginary) parts must be finite) this means that $\hat{f} \in L^1(\hat{G})$. Finally, we have, from Theorem 1.5.16 (Bochner's theorem) and the definition (1.5.1)

$$\int_{\hat{G}} \xi(x)\hat{f}(\xi) d\sigma_{\hat{G}}(\xi) = \int_{\hat{G}} \xi(x) d\mu_f(\xi) = f(x).$$

□

Corollary 1.5.20. *If $f \in L^1(G) \cap \mathcal{P}(G)$, then $\hat{f} \geq 0$.*

Proof. We know that $\hat{f} \in C_0(\hat{G})$, and from the proof of Theorem 1.5.19 (Fourier Inversion Theorem), we know that $\hat{f}(\xi) d\sigma_{\hat{G}}(\xi) = d\mu_f(\xi)$, and μ_f is positive, by Theorem 1.5.16 (Bochner's Theorem). □

When a Haar measure σ on G is given, the Haar measure that makes Theorem 1.5.19 true is called the **dual measure** of σ. If $\sigma_{\hat{G}}$ is the dual measure of σ, then $c^{-1}\sigma_{\hat{G}}$ is the dual measure of $c\sigma$, for $c > 0$. We will now always denote the dual measure of σ by $\sigma_{\hat{G}}$.

1.6. Plancherel Theorem

The Fourier inversion theorem is essentially half of a duality statement about "nice" functions $f : G \to \mathbb{C}$ determining precisely another "nice" function $\hat{f} : \hat{G} \to \mathbb{C}$. If we require the functions to be "nicer" then this relationship becomes exact and they determine each other completely.

Definition 1.6.1. *A **Banach space** X is a complete normed vector space. Examples of such spaces include $\mathbb{R}^n$, $\mathbb{C}^n$, the space of continuous functions $C(Y)$, L^p $(1 \leq p \leq \infty)$, among others.*

THEOREM 1.6.2 (Bounded Linear Extension). *Let X and Y be Banach spaces, Z a dense linear supspace of X, and T a bounded (continuous) linear operator from Z to Y. Then there is a unique bound (continuous) linear operator $\tilde{T}$ from X to Y such that $\tilde{T}$ is an extension of T (i.e. $T = \tilde{T}$ on Z). This satisfies $\|T\|_{Z,Y,op} = \|\tilde{T}\|_{X,Y,op}$.*

This can be found in [**Kre78**][Thm 2.7-11].

THEOREM 1.6.3 (Plancherel's Theorem). *The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$*

PROOF. Let $f \in L^1(G) \cap L^2(G)$. Then $f^* * f$ is continuous and in $L^1(G) \cap \mathcal{P}(G)$ by Lemmas 1.5.8 and 1.2.7 and $\mathcal{F}(f^* * f) = |\hat{f}|^2$ by Theorem 1.4.4, so by Theorem 1.5.19 (Fourier Inversion)

$$\begin{aligned}\int_G |f(x)|^2 d\sigma(x) &= \int_G f(x)\overline{f(-x)} d\sigma(x) \\ &= (f^* * f)(0) \\ &= \int_{\hat{G}} \mathcal{F}(f^* * f)(\xi)\xi(0) d\sigma_{\hat{G}}(\xi) \\ &= \int_{\hat{G}} |\hat{f}(\xi)|^2 d\sigma_{\hat{G}}(\xi),\end{aligned}$$

or

$$\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(G)}. \tag{1.6.1}$$

Thus $\mathcal{F}$ is an isometry in the L^2 norms from $L^1(G) \cap L^2(G)$ to its image. The space of all integrable simple functions on G is contained in $L^1(G) \cap L^2(G)$ and is dense in $L^2(G)$, and so $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$. Thus, by Theorem 1.6.2 $\mathcal{F}$ can be uniquely extended to a linear map from $L^2(G)$ to $L^2(\hat{G})$. This is an isometry (meaning it is unitary), as for $f \in L^2(G)$ and $f_n \in L^1(G) \cap L^2(G)$ such that $f_n \to f$ in $L^2(G)$, using (1.6.1),

$$\|f\|_{L^2(G)} = \lim_{n\to\infty} \|f_n\|_{L^2(G)} = \lim_{n\to\infty} \|\hat{f}\|_{L^2(G)} = \|\hat{f}\|_{L^2(G)}.$$

Now, suppose that $\psi \in L^2(\hat{G})$ but $\psi \notin \mathcal{F}(L^2(G))$. Since $\mathcal{F}$ is a continuous linear operator, this must mean that ψ is orthogonal to all $\hat{f} \in \mathcal{F}(L^1(G) \cap L^2(G))$, which is invariant under translations, L_y. Then by Theorem 1.4.4 for $f \in L^1(G) \cap L^2(G)$ and $x \in G$,

$$0 = \int_{\hat{G}} \mathcal{F}(L_x f)(\xi)\overline{\psi(\xi)} d\sigma_{\hat{G}}(\xi) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)\overline{\psi(\xi)} d\sigma_{\hat{G}}(\xi).$$

Since $\psi, \hat{f} \in L^2(\hat{G})$, $\hat{f}\overline{\psi} \in L^1(G)$, so $\hat{f}(\xi)\overline{\psi(\xi)} d\sigma_{\hat{G}}(x)$ represents an element of $M(\hat{G})$. From the injectivity of Proposition 1.5.3, $\hat{f}\psi = 0$ a.e. for all $f \in L^1(G) \cap L^2(G)$. Lemma 1.5.18 tells us there exist some $f \in C_c(G) \cap \mathcal{P}(G) \subseteq L^1(G) \cap L^2(G)$ such that $\hat{f} \geq 0$ everywhere and $\hat{f} > 0$ on some compact set. Thus, ψ must be zero a.e.

□

COROLLARY 1.6.4. *If G is compact and $\sigma(G) = 1$, then $\hat{G}$ is an orthonormal basis for $L^2(G)$.*

PROOF. We already know that $\hat{G}$ is an orthornormal set from Proposition 1.3.6. If $f \in L^2(G)$ is orthogonal to every $\xi \in \hat{G}$, then

$$0 = \int_G f(x)\overline{\xi(x)} d\sigma(x) = \hat{f}(\xi)$$

for all $\xi \in \hat{G}$, so $f = 0$, by Theorem 1.6.3 (Plancherel's Theorem). □

THEOREM 1.6.5 (Parseval's Formula). *For any $f, g \in L^2(G)$, we have*

$$\int_G f(x)\overline{g(x)}d\sigma(x) = \int_{\hat{G}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\sigma_{\hat{G}}(x).$$

PROOF. By Theorems 1.6.3 (Plancherel) and 1.4.4, we have

$$\begin{aligned} 4\int_G f(x)\overline{g(x)}d\sigma(x) &= \int_G |f(x)+g(x)|^2 - |f(x)-g(x)|^2 + i|f(x)+ig(x)|^2 - i|f(x)-ig(x)|^2 d\sigma(x) \\ &= \int_{\hat{G}} |\hat{f}(\xi)+\hat{g}(\xi)|^2 - |\hat{f}(\xi)-\hat{g}(\xi)|^2 + i|\hat{f}(\xi)+i\hat{g}(\xi)|^2 - i|\hat{f}(\xi)-i\hat{g}(\xi)|^2 d\sigma_{\hat{G}}(\xi) \\ &= 4\int_{\hat{G}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\sigma_{\hat{G}}(\xi). \end{aligned}$$

□

1.7. Pointryagin Duality

As we have shown in 1.3.5, $\hat{G}$ is a locally compact Abelian group, so it must have a dual $\hat{\hat{G}}$, with operation $\times$, and for $g \in L^1(\hat{G}) \cup L^2(\hat{G})$ and $\psi \in \hat{\hat{G}}$,

$$\hat{g}(\psi) = \int_{\hat{G}} g(\xi)\overline{\psi(\xi)}d\sigma_{\hat{G}}(\xi).$$

Each $x \in G$ defines a character on $\hat{G}$ by

$$\psi_x(\xi) = \xi(x), \tag{1.7.1}$$

which follows from Proposition 1.4.7 and the fact that $\xi(x)\eta^{-1}(x) = (\xi\eta^{-1})(x)$. We define the map $m : G \to \hat{\hat{G}}$ by

$$m(x) = \psi_x. \tag{1.7.2}$$

PROPOSITION 1.7.1. *For compact $K \subseteq G$ and $C \subseteq \hat{G}$ and $r > 0$, let*

$$N_G(K, r) = \{\xi \in \hat{G} : |1 - \xi(x)| < r, \ \forall x \in K\} \tag{1.7.3}$$

and

$$M_G(C, r) = \{x \in G : |1 - \xi(x)| < r, \ \forall \xi \in C\}. \tag{1.7.4}$$

These sets (and their translates) are open in their respective topologies. The family of all sets $N_G(K, r)$ forms a local basis of 1 in $\hat{G}$ and the collection of all their translates is a base for the topology of $\hat{G}$. The family of all sets $M_G(C, r)$ is a basis for 0 in G and the collection of all of their translates forms a base for the topology of G.

PROOF. Let $C \subseteq \hat{G}$ be compact, set $r > 0$, and let $x \in M_G(C, r)$. For each $\xi \in C$, there are neighborhoods U_ξ of ξ and W_ξ of x such that $|1 - \eta(y)| < r$ for $\eta \in U$ and $y \in W$. This follows from continuity (Proposition 1.4.7). Since C is compact, finitely many sets U_{ξ_n} cover C. If $V = \cap_{j=1}^n W_{\xi_j}$ (the intersection of the corresponding neighborhoods of x), we see that $V \subseteq M_G(C, r)$. Thus, every element of $M_G(C, r)$ has a neighborhood contained in $M_G(C, r)$, meaning this set is open. Since translation is continuous, the translates of these sets are open.

A similar proof works for $N_G(K, r)$, but we also have this by the way we define the compact convergence topology on $\hat{G}$ (see Definition 1.3.3). This definition also tells us that sets of the form (1.7.3) and their translates form a basis if the topology. Thus, all the elements of this basis which are neighborhoods of 1 form a local basis for 1. The elements of this local basis must be of the form

$$N_G(K, r, \eta) = \{\xi \in \hat{G} : |\eta(x) - \xi(x)| < r, \ \forall x \in K\},$$

for compact $K \subseteq G$, $\eta \in \hat{G}$ and $r > 0$. Suppose that $1 \in N_G(K, r, \eta)$, and let

$$s = \max\{|1 - \eta(x)| : x \in K\} < r.$$

Then for any $\xi \in N_G(K, r - s)$, we have that for all $x \in K$

$$|\eta(x) - \xi(x)| \leq |\eta(x) - 1| + |1 - \xi(x)| < r,$$

so $\xi \in N_G(K, r, \eta)$, and therefore $N_G(K, r - s) \subseteq N_G(K, r, \eta)$. Thus, the sets of the form (1.7.3) form a local basis of 1.

For the remaining part, we first show that $\{W - W : W \subseteq G \text{ open }\}$ form a local basis for $0 \in G$. Let $p : G \times G \to G$ be defined by $p(x, y) = x - y$. We give $G \times G$ the product topology, so $\{W \times U : U, W \subseteq G \text{ open}\}$ is a basis of the topology, and p is continuous. Now, let V be a neighborhood of 0. Then $p^{-1}(V) \subseteq G \times G$ is open and contains $(0, 0)$. Thus, there are some neighborhoods of 0, U and W, such that $U \times W \subseteq p^{-1}(V)$. We then see that $U \cap W$ is a neighborhood of 0, with $(U \cap W) - (U \cap W) \subseteq V$.

Now, let $U \subseteq G$ be a neighborhood of 0 and $V \subseteq G$ be a compact neighborhood of 0 (local compactness tells us this must exist). Let $W = U \cap V$, so $\overline{W}$ is a compact neighborhood of 0.

Now, let $f(x) = \frac{1}{\sqrt{\sigma(W)}} \mathbb{1}_W(x)$ and $g = f^* * f$. The function f is well defined, since W is a neighborhood of 0 with compact closure, so it has positive finite measure, and $f \in L^2(G)$. Thus $g \in C_0(G) \cap \mathcal{P}(G)$ (i.e. g is continuous, bounded, and of positive type), by Lemmas 1.2.7 and 1.5.8, and is (compactly) supported on $\overline{W - W}$, so $g \in L^1(G)$ as well. By Theorem 1.5.19 (Fourier Inversion Theorem), we have

$$\int_{\hat{G}} \hat{g}(\xi) d\sigma_{\hat{G}}(\xi) = g(0) = 1$$

and by Theorem 1.4.4, $\hat{g} = |\hat{f}|^2 \geq 0$. Thus, for $r \in (0, \frac{1}{2})$, there must exist some compact set $C \subseteq \hat{G}$ such that for

$$\int_C \hat{g}(\xi) d\sigma_{\hat{G}}(\xi) > \frac{2}{3}.$$

Now, suppose $x \in M_G(C, 1/3)$, so for $\xi \in C$, $|1 - \xi(x)| < \frac{1}{3}$ and so $\mathrm{Re}\xi(x) > \frac{2}{3}$. Thus

$$\mathrm{Re} \int_C \hat{g}(\xi)\xi(x) d\sigma_{\hat{G}}(\xi) \geq \int_C \hat{g}(\xi) \ \mathrm{Re}\xi(x) d\sigma_{\hat{G}}(\xi) > \frac{2}{3} \int_C \hat{g}(\xi) d\sigma_{\hat{G}}(\xi) > \frac{4}{9}$$

and

$$\Big| \int_{\hat{G} \setminus C} \hat{g}(\xi)\xi(x) d\sigma_{\hat{G}}(\xi) \Big| \leq \int_{\hat{G} \setminus C} \hat{g}(\xi) d\sigma_{\hat{G}}(\xi) < \frac{1}{3},$$

so, by Theorem 1.5.19 (Fourier Inversion) and the fact that g is real valued, for $x \in M_G(C, \frac{1}{3})$,

$$g(x) = \int_{\hat{G}} \hat{g}(\xi)\xi(x) d\sigma_{\hat{G}}(\xi) > \frac{1}{9}.$$

Since $g > 0$ only on $W - W$, we have $M_G(C, \frac{1}{3}) \subseteq W - W$. Thus, the family of sets $M_G(C, r)$ (for all compact $C \subseteq \hat{G}$ and $r > 0$) forms a local basis of 0. Translating each of these elements by $x \in G$ gives us a local basis of x, finishing our claim. $\square$

Lemma 1.7.2. *The map m defined by (1.7.2) is injective.*

Proof. Suppose, indirectly, that there is some $z \in G \setminus \{0\}$ such that $m(z) = m(0)$. Then $\psi_z(\xi) = \psi_0(\xi) = 1$ for all $\xi \in \hat{G}$. Thus, for all $f \in L^1(G)$ and $\xi \in \hat{G}$, by Theorem 1.4.4

$$\mathcal{F}(L_{-z}f)(\xi) = \int_G f(y-z)\overline{\xi(y)}d\sigma(y) = \int_G f(y)\overline{\xi(y)\xi(z)}d\sigma(y) = \hat{f}(\xi).$$

This means that $\mathcal{F}(L_{-z}f) = \hat{f}$ for all $L^1(G)$, so by Theorem 1.5.19 (Fourier Inversion), $L_{-z}f = f$ for all $f \in L^1(G) \cap \mathcal{B}(G)$.

Let $U, V \subseteq G$ be neighborhoods of 0 and z, respectively, such that $U \cap V = \emptyset$ (this can be done, as G is Hausdorff). We see that $V - z$ is a neighborhood of 0, and therefore so is $B = U \cap (V - z)$. Moreover

$$B \cap (B + z) = \Big(U \cap (V - z)\Big) \cap \Big((U + z) \cap V\Big) \subseteq U \cap V = \emptyset.$$

From the proof of Proposition 1.7.1, we know there is some neighborhood A of 0 such that $A - A \subseteq B$. Let K be a compact neighborhood of 0 (local compactness tells us this must exist), and $W = K \cap B$, so $\overline{W}$ is a compact neighborhood of 0.

Now, let $f(x) = \mathbb{1}_W(x)$ and $g = f^* * f$. The function $f \in L^2(G)$, so $g \in C_0(G) \cap \mathcal{P}(G)$ (i.e. g is continuous, bounded, and of positive type), by Lemmas 1.2.7 and 1.5.8, and is (compactly) supported on $\overline{W - W}$, so $g \in L^1(G) \cap \mathcal{B}(G)$. However, $\operatorname{supp}(g) \cap \operatorname{supp}(L_{-z}g) \subseteq B \cap (B+z) = \emptyset$, which is a contradiction. □

Proposition 1.7.3. *If $U \subset \hat{G}$ is open and non-empty, there exists some $f \in L^1(G) \cap L^2(G)$ such that $\hat{f} \neq 0$ and $\hat{f}(\xi) = 0$ outside of U.*

Proof. Let $K \subseteq U$ be compact, with $\sigma_{\hat{G}}(K) > 0$ and V be a compact neighborhood of 1 such that $KV = \{\xi\eta : \xi \in K, \eta \in V\} \subseteq E$. The proof that such a K and V can be chosen is similar to the proof that the sets $W - W$ form a basis, given in Proposition 1.7.1. Let $g = \mathbb{1}_K$, $h = \mathbb{1}_V$, and

$$p(\xi) = (g * h)(\xi) = \int_{\hat{G}} g(\eta)h(\xi\eta^{-1})d\sigma_{\hat{G}}(\eta).$$

We see that $\operatorname{supp}(p) \subseteq K + V \subseteq E$

$$\int_{\hat{G}} p(\xi)d\sigma_{\hat{G}}(\xi) = \int_{\hat{G}}\int_{\hat{G}} \mathbb{1}_K(\eta)\mathbb{1}_V(\xi\eta^{-1})d\sigma_{\hat{G}}(\eta)d\sigma_{\hat{G}}(\xi) = \sigma_{\hat{G}}(K)\sigma_{\hat{G}}(V) > 0,$$

so p is not identically 0 and $p \in L^2(\hat{G})$.

Since $g, h \in L^2(\hat{G})$, there must be some $a, b \in L^2(G)$ such that $\hat{a} = g$ and $\hat{b} = h$ by Theorem 1.6.3 (Plancherel's Theorem). By Hölder's inequality, we know that $\xi b \in L^2(G)$ for any $\xi \in \hat{G}$ and $ab \in L^1(G)$. Thus, by Theorems 1.4.4 and 1.6.5 (Parseval's Formula)

$$\int_G a(x)b(x)\overline{\xi(x)}d\sigma(x) = \int_{\hat{G}} g(\eta)h(\xi\eta^{-1})d\sigma_{\hat{G}}(\eta) = p(\xi),$$

so $p = \mathcal{F}(ab)$. Since $p \in L^2(G)$, by Theorem 1.6.3, $ab \in L^2(G)$, finishing the proof. □

As part of the proof of Proposition 1.7.3, we aslo showed the following:

Corollary 1.7.4. *If $f, g \in L^2(G)$, then $\mathcal{F}(fg)(\xi) = (\hat{f} * \hat{g})(\xi)$.*

Theorem 1.7.5 (Pontryagin duality). *The map is an isomorphism and a homeomorphism from G to $\hat{\hat{G}}$. In other words, G is the dual space of $\hat{G}$.*

PROOF. Let $x, y \in G$ and $\xi \in \hat{G}$. Then we have

$$\psi_{x+y}(\xi) = \xi(x+y) = \xi(x)\xi(y) = \psi_x(\xi)\psi_y(\xi) = (\psi_x \times \psi_y)(\xi),$$

so the $m(x+y) = m(x) \times m(y)$, so m is a homomorphism. Since m is injective, by Lemma 1.7.2, it is an isomorphism from G to its image in $\hat{\hat{G}}$.

Next we show that $m : G \to m(G)$ is a homeomorphism. By Proposition 1.7.1, sets of the form $N_{\hat{G}}(C, r)$ form a local basis at the identity in $\hat{\hat{G}}$ and sets of the form $M_G(C, r)$ form a local basis at 0 in G, where $C \subseteq \hat{G}$ are compact and $r > 0$. Note that this means that sets of the form $N_{\hat{G}}(C, r) \cap m(G)$ form a local basis of the identity in $m(G)$. By the definition of m, we have

$$\begin{aligned} m(M_G(C, r)) &= \{m(x) : x \in G, |1 - \xi(x)| < r \ \forall \xi \in C\} \\ &= \{\psi \in \hat{\hat{G}} : \psi = m(x) \text{ for some } x \in G \text{ and } |1 - \psi(\xi)| < r \ \forall \xi \in C\} \\ &= N_{\hat{G}}(C, r) \cap m(G). \end{aligned}$$

Thus, m and its inverse are continuous at 0, and since m is an isomorphism, the same result holds at an other point of G (or $m(G)$) by translation.

Thus, $m : G \to m(G)$ is a homeomorphism, so $m(G)$ is locally compact in the subspace topology $m(G)$ has as a subset of $\hat{\hat{G}}$. This means that for every $y \in m(G)$, there is an neighborhood V_y of y in $\hat{\hat{G}}$ such that the $m(G)$-closure of $V_y \cap m(G)$, which we will call K, is compact in $m(G)$. Since the inclusion map $i : H \to G$ is continuous, this means K is compact, and therefore closed, in $\hat{\hat{G}}$. Now, let $W = V_y \setminus (V_y \cap K)$. We see that $W \cap m(G) = \emptyset$.

If $m(G)$ is dense in $\hat{\hat{G}}$, then since W is open, W must be empty, so $V_y \subseteq K \subseteq m(G)$. Then we would have $m(G) = \cup_{y \in m(G)} V_y$, so $m(G)$ would be open. Since $m(G)$ is a subgroup of $\hat{\hat{G}}$, this must mean that every coset of $m(G)$ is open. As the complement of the union of all but one coset, this means that $m(G)$ is closed, which with density, would give us our claim. Thus, we need now only prove density.

Now suppose, indirectly, that $m(G)$ is not dense in $\hat{\hat{G}}$. Then, since $m(G)$ is closed, by Proposition 1.7.3, we know there is some $f \in L^1(\hat{G}) \cap L^2(\hat{G})$ such that $\hat{f}$ is not identically zero, but $\hat{f}(\psi) = 0$ for all $\psi \in m(G)$. Then, for all $x \in G$,

$$\int_{\hat{G}} f(\xi)\xi^{-1}(x) d\sigma_{\hat{G}}(\xi) = \int_{\hat{G}} f(\xi)\overline{\psi_x(\xi)} d\sigma_{\hat{G}}(\xi) = \hat{f}(\psi_x) = 0.$$

Since $f(\xi)d\sigma_{\hat{G}}(\xi)$ gives a complex Radon measure of finite variation, injectivity from Proposition 1.5.3 tells us that $f = 0$, and so $\hat{f} = 0$, which is a contradiction, so $m(G)$ must be dense. Our claim now follows. $\square$

1.8. Consequences of Duality

COROLLARY 1.8.1. *A locally compact Abelian group G is compact if and only if $\hat{G}$ is discrete. G is discrete if and only if $\hat{G}$ is compact.*

PROOF. This immediately follows from combining Theorem 1.4.3 with Theorem 1.7.5. $\square$

THEOREM 1.8.2 (Fourier Inversion Theorem 2). *If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, then*

$$f(x) = \hat{\hat{f}}(-x) \tag{1.8.1}$$

for x almost everywhere in G, i.e.

$$f(x) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)d\sigma_{\hat{G}}(\xi), \quad x \text{ a.e.} \tag{1.8.2}$$

If f is continuous, (1.8.2) *holds for all $x \in G$.*

PROOF. For all $\xi \in \hat{G}$, we have

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}d\sigma(x) = \int_G f(x)\xi(-x)d\sigma(x) = \int_G f(-x)\xi(x)d\sigma(x).$$

Defining $d\mu_{\hat{f}}(x) = f(-x)d\sigma(x)$, so $\mu_{\hat{f}} \in M(\hat{G})$, we see (considering the definition of $\mathcal{B}(\hat{G})$ and (1.5.1)) that $\hat{f} \in \mathcal{B}(\hat{G}) \cap L^1(\hat{G})$. By Theorem 1.5.19, we know that for all $\xi \in \hat{G}$

$$\hat{f}(\xi) = \int_G \hat{\hat{f}}(x)\xi(x)d\sigma(x).$$

Thus, $f(x) = \hat{\hat{f}}(-x)$ almost everywhere. Since $\hat{\hat{f}}$ is continuous, if f is continuous, they must be equal everywhere. □

THEOREM 1.8.3 (Uniqueness Theorem). *If $\mu, \nu \in M(G)$ and $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$. In particular, if $f, g \in L^1(G)$ and $\hat{f} = \hat{g}$, then $f = g$.*

PROOF. Proposition 1.5.3 now tells us, replacing $\hat{G}$ with G (which we may now do, as we have $G = \hat{\hat{G}}$, that $M(G)$ is mapped injectively to the the space of bounded continuous functions on $\hat{G}$, so μ is completely determined by the function $\phi_\mu(\xi) = \hat{\mu}(\xi^{-1})$. □

CHAPTER 2

Appendix

If I couldn't find a good reference for results, and/or I am using them but perhaps am not covering them in class, they are here.

PROPOSITION 2.0.1. *Let (X, μ) be a measure space and $f \in L^p(X)$ for some $p \in [1, \infty)$. Then* $\operatorname{supp}(f)$ *is σ-finite.*

PROOF. Let $g(x) = |f(x)|^p$, so that g is integrable. Let $A_0 = [1, \infty)$ and for $n \in \mathbb{N}$, $A_n = [\frac{1}{n+1}, \frac{1}{n})$. For each $n \in \mathbb{N}_0$, we have

$$\frac{\mu\big(g^{-1}(A_n)\big)}{n+1} \leq \int_{g^{-1}(A_n)} g(x) d\mu(x) \leq \int_X g(x) d\mu(x) < \infty. \tag{2.0.1}$$

Since g and f have the same support, we see that $\operatorname{supp}(f) = \cup_{n=0}^{\infty} g^{-1}(A_n)$, so the support of f is σ-finite. $\square$

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